ANALYSIS QUALIFYING EXAM

JANUARY 7, 2000 1:00–5:00 P.M.

Section A Answer 10 of the 12.

A1. Give an example of each of the following or state that no such example exists.

(a) A countably infinite set with an uncountable number of cluster points.

(b) A countably infinite set with one cluster point.

(c) A countably infinite set with no cluster points.

(d) A sequence of continuous functions whose limit is not continuous.

A2.

(a) Define *compact* and *sequentially compact.*

(b) Prove that a compact set is sequentially compact.

A3. For each of the functions below, determine if the limit as $(x, y) \to (0, 0)$ exists. Defend your claim.

(a)

$$f(x,y) = \frac{\sin x \sin y}{x^2 + y^2}$$

(b)

$$f(x,y) = \frac{x^3 - y^3}{x^2 + y^2}$$

(c)

$$f(x,y) = \frac{x^4 + y^2}{2x^4 + y^2}$$

A4. Find all local extrema of the function $f(x, y) = \sin x + \cos y$.

A5. Evaluate the integral

$$\int_0^2 \int_0^{\sqrt{4-x^2}} \sin\left(x^2 + y^2\right)\, dy\, dx$$

A5.

(a) Let f be an extended real-valued function on $\mathbb{R}$ such that $\{x\colon f(x) > r\}$ is measurable for all rational r. Must f be measurable? Prove your answer.

(b) Let f be a measurable function and E be a Borel set. Must $f^{-1}[E]$ be measurable?

A6.

(a) Let m^* be the (Lebesgue) outer measure. Construct, if possible, a sequence $\{E_n\}$ of disjoint (not necessarily measurable) sets such that $m^*\left(\bigcup_{n=1}^{\infty} E_n\right) < \sum_{n=1}^{\infty} m^*(E_n)$.

(b) Construct a sequence $\{F_n\}$ of (not necessarily measurable) sets such that $F_{n+1} \subseteq F_n$, $m^* F_n < \infty$ for all n, and $m^*\left(\bigcap_{n=1}^{\infty} F_n\right) < \lim_{n\to\infty} m^* F_n$.

A7.

(a) Let f and g be integrable functions on $[0,1]$ such that $f \geq g$ a.e. Suppose that $\int_0^1 f = \int_0^1 g$. Must it follow that $f = g$ a.e.? Prove your answer.

(b) Let f be an integrable function on $\mathbb{R}$. For each $n = 1, 2, \ldots$, let f_n be defined by

$$f_n(x) = \begin{cases} f(x), & \text{if } x \in [-n, n] \text{ and } |f(x)| \leq n; \\ 0, & \text{otherwise.} \end{cases}$$

Does $\int_{\mathbb{R}} |f - f_n| \to 0$ as $n \to \infty$? Prove your answer.

A8. Give an example, if possible, of each of the following. Prove your answers.

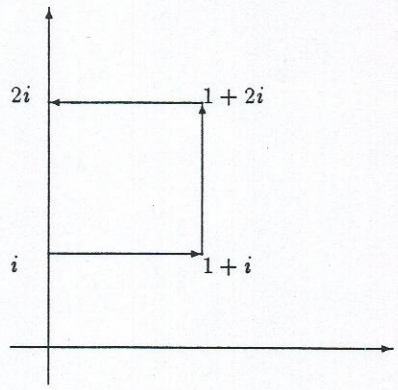
(a) A non-zero bounded linear functional on $L^p[0,1]$.

(b) An unbounded sequence $\{f_n\}$ in $L^p[0,1]$, $1 \leq p \leq \infty$, such that $\|f_n - f\|_p \to 0$ for some $f \in L^p[0,1]$ but $f \not\to f$ a.e.

(c) A function in $L^p[0,\infty)$, $1 \leq p < \infty$, that is not bounded and nonzero everywhere.

A9.

(a) Use the definition of a complex integral to evaluate $\int_C z^2\,dz$ if C is the curve from i to $2i$ on the right side.



(b) Is the integral in part (a) independent of the path? If so, calculate it in another way than you did in part (a). Justify your answers.

A10. Prove that if $f(z)$ is an entire function and $M(\rho) = \max\{|f(z)| : |z| = \rho\}$, is such that $M(\rho) \le L\rho^k$, then $f(z)$ is a polynomial of degree at most k.

A11. Let f be entire and suppose that $\operatorname{Re} f(z) \le \operatorname{Im} f(z)$ for all $z \in \mathbb{C}$. Prove that f is constant.

A12.

(a) State Rouche's Theorem.

(b) Let $p(z) = z^5 + 6z^3 + 2z + 10$. Show that all five zeros of $p(z)$ lie in the set $\{z : 1 < |z| < 3\}$.

Section B Select one of the 2 parts.

Part I Answer 2 of the 3.

B1. Let $\mu << m$, where m is Lebesgue measure on $\mathbb{R}$. Define $F(x) = \mu(-\infty, x]$. Show that F is absolutely continuous.

B2. Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be finite measure spaces. Discuss the development of a product measure $\mu_1 \times \mu_2$ on $\Sigma_1 \times \Sigma_2$ and state Fubini's theorem.

B3. Let $(X, \mathcal{M})$ be a measure space, and let μ, ν be two positive measures on $\mathcal{M}$ such that $\mu(X) + \nu(X) < \infty$. Assume $\mu << \nu$ and let g be the Radon-Nikodym derivative of μ with respect to ν. Show that $L^p(\nu) \subset L^1(\mu)$ if and only if $g \in L^q(\nu)$, $\frac{1}{p} + \frac{1}{q} = 1$.

Part II Answer 2 of the 3.

B4. Suppose $\langle u_n(z) \rangle$ is a sequence of harmonic functions that converges uniformly on all compact subsets of a domain $\mathcal{D}$ to a function $u(z)$. Then prove that $u(z)$ is harmonic throughout $\mathcal{D}$.

B5. Given a set of real numbers $0 \leq \theta_1 < \theta_2 < \cdots < \theta_n < 2\pi$, construct a function $f(z)$ satisfying

1. $f(z)$ is analytic in $|z| < 1$ and
2. The only singular points of $f(z)$ on the unit circle are at $e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n}$.

Prove these assertions for your choice of $f(z)$.

B6. Prove Weierstrass' Theorem: Given any complex sequence having no finite limit point, there exists an entire function that has zeros at these points and only these points.