Analysis Qualifying Exam January 19, 2001, 1:00-5:00

## Section A. Do both parts.

**Part 1** Do (only) *five* of the six problems.

1. Describe the development of Lebesgue measure and the Lebesgue integral, starting with definition of outer measure and concluding with the important convergence theorems. State carefully the pertinent definitions and theorems.

2. (a) Give an example of a function that is not Lebesgue integrable on $[0, 1]$ and justify your answer.

   (b) Give an example of a Lebesgue integrable function on $[1, \infty)$ and prove it using one of the convergence theorems.

3. Suppose $f$ is Lebesgue measurable on $\mathbb{R}$. Is it true that there must exist a sequence $\{E_n\}$ of subsets of $\mathbb{R}$ whose union is $\mathbb{R}$ such that each $E_n$ has finite measure and $f$ is bounded on each $E_n$? Give a proof if true and counterexample if not.

4. Let

$$f(x) = \begin{cases} 2\sqrt{x}, & \text{if } 0 \leq x \leq 1/4; \\ -2x + 3/2, & \text{if } 1/4 < x \leq 1. \end{cases}$$

   (a) Is the function $f$ absolutely continuous on $[0, 1]$? How do you know?

   (b) Is $f$ of bounded variation on $[0, 1]$? How do you know?

   (c) If the answer in b) is yes, find the total variation of $f$ on $[0, 1]$.

5. (a) Define what is meant by $L^p[0, 1]$, $1 \leq p < \infty$.

   (b) What does the Minkowski inequality for $L^p$ say?

   (c) Given $\frac{1}{p} + \frac{1}{q} = 1$ and $g \in L^q$, define $F$ on $L^p[0, 1]$ by

$$F(f) = \int_0^1 fg.$$

   What can you say about the function $F$?

6. For each positive integer $n$ let $f_n(x) = \sum_{k=1}^{n} \frac{1}{k} \sin kx$ for $x \in [0, 2\pi]$.

(a) Is it true that $f_n \in L^2[0, 2\pi]$ for each $n$?

(b) Is the sequence $\{f_n\}$ a Cauchy sequence in $[0, 2\pi]$? Justify your answer.

(c) If the answer in b) is yes, is there a function $f$ in $L^2[0, 2\pi]$ so that $f(x) = \sum_{n=1}^{\infty} \frac{1}{n} \sin nx$ a.e.? What theorem would tell you this?

**Part 2** Do (only) *five* of the six problems.

1. (a) Show that $h(z) = x^3 + 3xy^2 - 3x + i(y^3 + 3x^2y - 3y)$ is differentiable on the coordinate axes but is nowhere analytic.

   (b) Show that if the anlytic function $w = f(z)$ maps a domain $D$ onto a portion of a line, then $f$ must be constant throughout $D$.

2. (a) Prove that there exists no function $F(z)$ analytic in the annulus $D : 1 < |z| < 2$ such that $F'(z) = \frac{1}{z}$ for all $z$ in $D$.

   (b) Define a branch of $(z^2 - 1)^{1/2}$ that is analytic in the exterior of the unit circle, $|z| > 1$.

3. Compute $\int_C \frac{z+i}{z^3 + 2z^2} dz$ where $C$ is

   (a) the circle $|z| = 1$ traversed once counterclockwise;

   (b) the circle $|z + 2 - i| = 2$ traversed once counterclockwise.

4. For each of the following, construct a function $f$ analytic in the plane except for isolated singularities that satisfies the given conditions and justify your answer.

   (a) $f$ has a zero of order 2 at $z = i$ and a pole of order 5 at $z = 2 - 3i$;

   (b) $f$ has a simple zero at $z = 0$ and an essential singularity at $z = 1$;

   (c) $f$ has a removable singularity at $z = 0$, a pole of order 6 at $z = 1$, and an essential singularity at $z = i$.

5. Compute

$$\oint_{|z|=5} \left[ ze^{3/z} + \frac{\cos z}{z^2(z-\pi)^3} \right] dz$$

using the Cauchy residue theorem.

6. (a) State Rouche's Theorem.

   (b) Using Rouche's Theorem prove that every polynomial of degree $n$ has $n$ zeros.

## Section B Do (only) *two* of the three problems.

1. (a) Let $(X, \rho)$ be a metric space; $E \subseteq X$; and let $f : E \to \mathbb{R}$ be a uniformly continuous function. Show that there is a unique extension of $f$ to a uniformly continuous function $g$ from the closure $\overline{E}$ of $E$ to $\mathbb{R}$.

   (b) If we assume merely the continuity of $f$ above, can we still conclude that there is a continuous extension? Explain your answer.

2. Let $X$ be a normed linear space.

   (a) Let $x \in X$. Show that there is a bounded linear functional $f$ on $X$ such that $f(x) = \|f\| \, \|x\|$.

   (b) Explain the difference between the weak topology and the weak* topology on the dual $X^*$ of $X$.

   (c) Give an example each of a typical basic open set containing $0 \in X^*$ in the weak topology and the weak* topology on $X^*$.

3. (a) Let $(X, \mathcal{B}, \mu)$ be a measure space, and $\nu$ is a measure on $(X, \mathcal{B})$. Under what conditions can we conclude that there is a measurable function $f$ such that $\nu E = \int_E f \, d\mu$ for all $E \in \mathcal{B}$?

(b) Let $X = [0,1] \cup \{\infty\}$; $\mathcal{B}$ be the $\sigma$-algebra of sets $E \subseteq X$ such that $E \sim \{\infty\}$ is a Lebesgue measurable subset of $[0,1]$; and let $m$ be the Lebesgue measure restricted to $[0,1]$. Define $\mu$ and $\nu$ on $(X, \mathcal{B})$ by

$$\mu E = \begin{cases} mE & \text{if } \infty \notin E \\ \infty & \text{if } \infty \in E \end{cases},$$

and $\nu E = m(E \sim \{\infty\})$. Discuss the mutual singularity and absolute continuity between $\mu$ and $\nu$.

(c) Discuss the existence of a function as in part (a) that works for either $\mu$ with respect to $\nu$ or the other way. Explain your conclusions.

(d) Discuss the existence of a Lebesgue decomposition of one measure with respect to the other. Explain your conclusions.