Analysis Qualifying Exam

January 18, 2002
Central Michigan University

Section A. Do both parts.

Part 1. Do (only) *five* of the six problems.

1. Let $\chi_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \sim \mathbb{Q} \end{cases}$ be the characteristic function of the set of rational numbers $\mathbb{Q}$ and let

$$\mathcal{F} = \left\{ f \in \mathbb{R}^{\mathbb{R}} : f \text{ is differentiable at exactly three points} \right\}$$

be the collection of real–valued functions on $\mathbb{R}$ that are differentiable at exactly three points.

(a) Prove that $x^2\chi_{\mathbb{Q}}$ is differentiable at $x = a \quad \Leftrightarrow \quad a = 0$.

(b) Give an explicit definition of a specific function belonging to $\mathcal{F}$.

(c) Determine the cardinality of $\mathcal{F}$.

2. Let λ^* denote Lebesgue outer measure on $\mathbb{R}$. Do not use the fact that λ^* is countably subadditive unless you prove it.

(a) **Prove:** $\lambda^*(\mathbb{Q}) = 0$.

(b) **Prove:** If $\lambda^*(Z) = 0$, then there exists a $\mathcal{G}_\delta$ set H such that $Z \subseteq H$ and $\lambda^*(H) = 0$.

(c) Give a brief justification of the following. (You may quote any of the standard versions of the Baire category theorem without proof.) There exist sets F and Z such that F is first category in $\mathbb{R}$, $\lambda^*(Z) = 0$, and $\mathbb{R} = F \cup Z$.

3. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be such that f is unbounded on every set of positive measure. Using Lusin's theorem, or another method of your choice, prove that f is a non–measurable function.

(b) Show that the converse of the statement in (a) is false.

4. Let $f : [0, 1] \to \mathbb{R}$ be measurable with $f \geq 0$ and, for each $n \in \mathbb{N}$, let $E_n = \{x \in [0, 1] : n - 1 \leq f(x) < n\}$. Prove that

$$f \in \mathcal{L}^1[0, 1] \iff \sum_{n=1}^{\infty} [n \cdot \lambda(E_n)] < \infty,$$

where λ is Lebesgue measure on $\mathbb{R}$.

5. Let $f : [0, 1] \to \mathbb{R}$ be absolutely continuous on $[0, 1]$ and, for each $E \subseteq [0, 1]$, let $f[E] = \{f(x) : x \in E\}$ be the image of E under f.

(a) **Prove:** If $Z \subseteq [0, 1]$ has measure zero, then $f[Z]$ has measure zero.

(b) **Prove:** If $Z \subseteq [0, 1]$ is measurable, then $f[Z]$ is measurable.

(c) For each of the following four classes of functions defined on $[0, 1]$, state whether (a) or (b) (or both) is true for that class. (You do not have to justify your answers.)

(i) **continuous**
(ii) **Lipschitz continuous**
(iii) **bounded variation**
(iv) **continuous and bounded variation**

6. Prove **TWO** of the following. You may use basic properties of integration and quote standard theorems without proof.

(a) $\int_0^1 \chi_E$ exists as a Riemann integral, where χ_E is the characteristic function of E and E is any subset of a closed measure zero set.

(b) $\int_0^1 f$ does not exist as a Riemann integral for some function f that is a pointwise limit of step functions on $[0, 1]$.

(c) $\int_0^1 \left[\frac{1}{x}\sin\left(\frac{1}{x}\right)\right] dx$ exists and is finite as an improper Riemann integral, but not as a Riemann integral or as a Lebesgue integral.

Part 2. Do (only) *five* of the six problems.

1. (a) State the Cauchy–Riemann equations for a complex function $f(z) = u(x, y) + iv(x, y)$.

(b) Consider the function

$$f(z) = \begin{cases} \dfrac{(\overline{z})^2}{z}, & \text{when } z \neq 0; \\ 0, & \text{when } z = 0. \end{cases}$$

Does this function satisfy the Cauchy–Riemann equations at the origin $z = (0, 0)$?

(c) Does the function in part (b) have a derivative at $(0, 0)$? Does your answer contradict a theorem relating differentiability and the Cauchy–Riemann equations? Explain.

2. (a) What do we mean by saying a function is *analytic* at a point? On a domain?

   (b) Explain what is meant by *a branch cut* for a complex function f.

   (c) Determine a branch cut and therefore a domain of analyticity for the function $f(z) = \log(3z - i)$. Compute $f'(z)$.

3. (a) What is meant by a *contour*?

   (b) Evaluate the integral $\int_\Gamma \frac{dz}{2z-1}$, where Γ is the circle of radius one centered at $z = \frac{1}{2}$, oriented positively. Use a parametrization of Γ and evaluate the integral directly.

   (c) Evaluate the integral of the same function as in part (b), but around the circle $|z| = 1$. You may use any pertinent theorems.

4. (a) Discuss what you know about the relationship between analyticity of a function and its expansion as a power series.

   (b) Suppose f is continuous in a bounded open disk D and on its boundary. Suppose further that $\int_\Gamma f(z) = 0$ for every closed contour Γ inside D. Must f be expandable in a Taylor series around each point of D? Justify your answer.

   (c) Suppose for the function in part (b) it is known that $|f(z) - 1| < 1$ for all z on the boundary of D. Prove that f has no zero in D.

5. (a) Find the Laurent series expansion for $f(z) = \frac{1}{z+z^2}$ in the domain $0 < |z+1| < 1$.

   (b) State the Cauchy Residue Theorem and use it to evaluate the following integral:

$$\int_{|z|=3} \frac{e^z}{z(z-2)^3}\, dz$$

6. Show that if f is a continuous function from the complex numbers into the complex numbers which is analytic off the interval $[-1, 1]$, then f is an entire function.

Section B. Do (only) *two* of the three problems.

1. (a) Write the simplest entire function with a zero of order 5 at 0 and simple zeros at odd positive integers as an infinite product. Explain your answer, including why the product converges.

   (b) Show that $\prod_{n=1}^{\infty}(1+z_n)$ converges absolutely iff $\prod_{n=1}^{\infty}(1+|z_n|)$ converges.

2. (a) What does Runge's theorem say?

   (b) Let K be a compact subset of an open set $G \subseteq \mathbb{C}$ and let $E \subseteq \mathbb{C} - G$. Suppose R is a rational function with a simple pole at $a \in \overline{E}$. Show that for each $\epsilon > 0$ there is a rational function R_1 with a simple pole in E such that $|R(z) - R_1(z)| < \epsilon$ for all $z \in K$. Can you extend this to more than one pole? Explain.

3. (a) Let G be a region in $\mathbb{C}$ and let $u : G \to \mathbb{R}$ be a continuous function which has the mean value property. Prove that u is harmonic on G.

   (b) Prove that a non–constant harmonic function on a region is an open map.