Analysis Qualifying Exam: January 7, 2010

MTH 632: Provide complete solutions to 5 of the 6 questions.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Show that $f^{-1}(E)$ is a Borel set whenever E is a Borel set. Explain why this shows that f is a measurable function.

2. Let $\langle f_n \rangle$ be a sequence of nonnegative functions. Show
$$\int \underline{\lim}\, f_n \le \underline{\lim} \int f_n.$$

3. Let $\langle f_n \rangle$ be a sequence of measurable functions such that $f_n \to 0$ in measure, and suppose that for all $\epsilon > 0$,
$$\sum_{n=1}^{\infty} m(\{x : |f_n(x)| > \epsilon\}) < \infty.$$
Show that $f_n \to 0$ *a.e.*

   Hint: Consider the sets $A(\epsilon) = \{x : \forall k,\ \exists n \ge k,\ |f_n(x)| > \epsilon\}$ and $A_n(\epsilon) = \{x : |f_n(x)| > \epsilon\}$.

4. Suppose f is an integrable function on $\mathbb{R}$ then
$$\lim_{t \to 0} \int_{-\infty}^{\infty} |f(x+t) - f(x)|\, dx = 0.$$

5. (a) The Fundamental Theorem of Calculus tells us that if f is continuous on $[a, b]$ then $\dfrac{d}{dx} \displaystyle\int_a^x f(t)\, dt = f(x)$. Does this still hold if f is simply integrable? If yes prove your answer. If not, what conditions on f are necessary for it to hold.

   (b) The Fundamental Theorem of Calculus also tells us that if $f'(x)$ is continuous then $\displaystyle\int_a^x f'(t)\, dt = f(x) - f(a)$. Is there a wider class of functions for which this holds? Prove your answer.

6. Let a sequence $\langle g_n \rangle$ in $L^q[0,1]$, $1 < q < \infty$ have the property that $\left| \displaystyle\int_0^1 f g_n \right| \le ||f||_p$ for all n and all $f \in L^p[0,1]$, $\frac{1}{p} + \frac{1}{q} = 1$. Answer the following questions with a proof for a yes answer, and a counterexample for a no answer.

   (a) Does it follow that $\langle ||g||_q \rangle$ is bounded?

   (b) Must there be a subsequence $\langle g_{n_k} \rangle$ of $\langle g_n \rangle$ and a $g \in L^q[0,1]$ such that
$$||g_{n_k} - g||_q \to 0 \text{ as } k \to \infty?$$

MTH 636: Provide complete solutions to 6 of the 7 questions.

1. Let $f(z) = \begin{cases} \dfrac{z^5}{|z|^4} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0. \end{cases}$

   Show that the Cauchy-Riemann equations hold at $z = 0$, but $f$is not differentiable at $z = 0$.

2. If f is analytic in the annulus $1 \leq |z| \leq 2$ and $|f(z)| \leq 3$ on $|z| = 1$ and $|f(z)| \leq 12$ on $|z| = 12$, prove that $|f(z)| \leq 3|z|^2$ for $1 \leq |z| \leq 2$.

   Hint: Consider $\frac{f(z)}{3z^2}$.

3. Let g be a continuous on the real interval $[-1, 2]$, and for each complex number z define

$$F(z) := \int_{-1}^{2} g(t) \sin(zt)\, dt.$$

   Prove that F is entire, and find its power series around the origin. Also, prove that for all z

$$F'(z) := \int_{-1}^{2} t g(t) \cos(zt)\, dt.$$

4. Find the Laurent series for the function

$$f(z) = \frac{1}{(z-1)(z-2)}$$

   in each of the following domains:

   (a) $|z| < 1$ (b) $1 < |z| < 2$ (c) $2 < |z|$

5. Does there exist a function $f(z)$ analytic in $|z| < 1$ and satisfying

   $f\left(\frac{1}{2}\right) = \frac{1}{2}$, $f\left(\frac{1}{3}\right) = \frac{1}{2}$, $f\left(\frac{1}{4}\right) = \frac{1}{4}$, $f\left(\frac{1}{5}\right) = \frac{1}{4}$, $\ldots,$

   $f\left(\frac{1}{2n}\right) = \frac{1}{2n}$, $f\left(\frac{1}{2n+1}\right) = \frac{1}{2}$, $\ldots$?

   Justify your answer.

6. Prove that the equation $z + 3 + 2e^z = 0$ has precisely one root in the left half-plane.

7. Using the method of residues verify that $\displaystyle\int_{-\pi}^{\pi} \frac{1}{1 + \sin^2\theta}\, d\theta = \pi\sqrt{2}.$