Analysis Qualifying Examination January 13, 2012

MTH 632: Provide complete solutions to **only** five of the six problems.

1. Let E be a measurable set of finite outer measure. Then, show that, for each $\epsilon > 0$, there is a finite disjoint collection of open intervals $\{ I_k \}_{k=1}^{n}$ for which if $\cup_{k=1}^{n} I_k = O$, then $m^*(E \sim O) + m^*(O \sim E) < \epsilon$.

2. Let the function f be defined on a measurable set E. Show that f is measurable if and only if for each Borel set A, $f^{-1}(A)$ is measurable.

3. Let $\{f_n\}$ be a sequence of nonnegative measurable functions on E. Show that

$$\int\limits_E liminf f_n \leq liminf \int\limits_E f_n$$

4. Let $f \in L^1[0, \infty)$ and define

$$g(y) = \int\limits_0^\infty f(x) cos(xy) dx.$$

Show that
 (i) g is a bounded function, and
 (ii) g is a continuous function of y on all of $\mathbb{R}$.

5. Let f and g be absolutely continuous functions on $[a, b]$. Show that
 (i) fg, their product, is absolutely continuous, and
 (ii)

$$\int_a^b f(t)g'(t)dt = f(b)g(b) - f(a)g(a) - \int_a^b f'(t)g(t)dt$$

6. Let $\{f_n\}$ be a sequence of functions in $L^2[a, b]$. Suppose $f \in L^2[a, b]$ is such that $\lim\limits_{n\to\infty} \parallel f_n - f \parallel_2 = 0$. Show that

 (i) $\displaystyle\int_a^b f^2(t)dt = \lim_{n\to\infty} \int_a^b f_n^2(t)dt$, and
 (ii) $\displaystyle\int_a^x f(t)dt = \lim_{n\to\infty} \int_a^x f_n(t)dt$ for $a \leq x \leq b$.

MTH 636: Provide complete solutions to **only** 5 of the 6 problems.

1. (a) Describe the range of $f(z) = -\frac{1}{2}z^3$ defined on $\{ z = x + iy : |z| < 1, \ x > 0, \ y > 0 \}$.
   (b) Prove that $\lim_{z \to i} z^2 = -1$.

2. Find a harmonic conjugate of $u = e^x \sin y$.

3. Compute
   (a)
   $$\int_C \frac{1}{z} \, dz,$$
   where C is defined by $x^2 + 4y^2 = 1$, traversed once counterclockwise.
   (b)
   $$\int_C |z| \, dz,$$
   where C is the line segment with the initial point $(-1 - i)$ and the final point $(1 + i)$.

4. Prove that for any z such that $|z| < 1$,
   $$\left| \frac{\left(\frac{1}{2} + \frac{1}{3}i\right) - z}{1 - \left(\frac{1}{2} - \frac{1}{3}i\right) z} \right| < 1.$$

5. A doubly periodic function is a function defined at all points on the complex plane and having two "periods", which are complex numbers u and v, where u and v are not real multiples of each other. That u and v are periods of a function f means that
   $$f(z) = f(z + u) = f(z + v),$$
   for all values of the complex number z. Give an example of a non constant doubly periodic complex valued function on $\mathbb{C}$. Is it possible to find a non constant analytic example?

6. Recall Jordan's lemma: If $m > 0$ and P/Q is the quotient of two polynomials such that
   $$\textit{degree } Q \geq 1 + \textit{degree } P,$$
   then
   $$\lim_{\rho \to \infty} \int_C e^{imz} \frac{P(z)}{Q(z)} \, dz = 0,$$
   where C is the upper half-circle of radius ρ. Prove Jordan's lemma directly (without quoting the lemma itself) in the case that $m = 1$, $P = 1$ and $Q = z^3$.