Analysis Qualifying Examination January 3, 2013

MTH 632: Provide complete solutions to **only** 5 of the 6 problems.

1. Let I be an interval and $f : I \to \mathbf{R}$ be increasing. Show that f is measurable by first showing that, for each natural number n, the strictly increasing function $x \mapsto f(x) + x/n$ is measurable, and then taking pointwise limits.

2. Let $f \in L^1[0, \infty)$ and define

$$g(y) = \int_0^\infty f(x) \cos(xy)\, dx$$

Show that
(a) g is a bounded function, and
(b) g is a continuous function of y on all of $\mathbb{R}$

3. Let g be a nonnegative integrable function over E and suppose $\{f_n\}$ is a sequence of measurable functions on E such that for each n, $|f_n| \leq g$ a.e. on E. Show that

$$\int_E \liminf f_n \leq \liminf \int_E f_n \leq \limsup \int_E f_n \leq \int_E \limsup f_n.$$

4. Let f be a real-valued integrable function defined on $\mathbb{R}$. Let $\langle E_n \rangle$ be a sequence of measurable sets such that $\lim\limits_{n\to\infty} m(E_n) = 0$. Show that $\lim\limits_{n\to\infty} \int_{E_n} f\, dm = 0$.

5. (a) Let f be a continuous function on $[0, 1]$ that is absolutely continuous on $[\epsilon, 1]$ for each $0 < \epsilon < 1$.

   (b) Show that f may not be absolutely continuous on $[0, 1]$.

   (c) Show that f is absolutely continuous on $[0, 1]$ if it is increasing.

6. (a) If g is a function in L^q, define a function F on L^p where $\frac{1}{q} + \frac{1}{p} = 1$, by the equation

$$F(f) = \int fg.$$

   Show that F is a bounded linear function on L^p.

   (b) If $f \in L^1$ and $g \in L^\infty$, then show that

$$\int |fg| \leq \|f\|_1 \|g\|_\infty .$$

MTH 636: Provide complete solutions to **only** 6 of the 7 problems.

1. Find the set of points z at which $f(z) = |z^3|$ is differentiable. Justify your answer.

2. Show that $4e, e = 2.718...$ is an upper bound for $|\int_C z^2 e^z dz|$ where C is the straight line contour joining the points $1+i$ and $1-i$.

3. If f is analytic for $|z| \leq 1$ then show that
$$\frac{1}{\pi}\int_{|z|\leq 1} f(x+iy)dxdy = f(0).$$

4. Find the series representation of
$$f(z) = \frac{1}{(z-1)(z-2)}$$
in the following two annuli centered at 0: (i) $1 < |z| < 2$, (ii) $2 < |z| < +\infty$.

5. Use Rouche's Theorem to show that all zeros of the complex-valued polynomial
$$p(z) = z^6 - 4z^2 + 11$$
lie in the annulus $1 < |z| < 2$.

6. If $f(z) = g(z)/h(z)$ where g and h are analytic at $z = \alpha$ but h has a simple zero at $z = \alpha$, then show that the residue of f at α is given by
$$a_{-1} = \frac{g(\alpha)}{h'(\alpha)}.$$

7. Show that
$$\int_{-\infty}^{+\infty} \frac{x \sin x}{1+x^2} dx = \frac{\pi}{e},$$
where the integral converges in the sense of Cauchy.