Analysis Qualifying Examination January 10, 2014

MTH 632: Provide complete solutions to **only** 5 of the 6 problems.

1. Prove that if $E \subset [a, b]$ and $m^*(E) = 0$ then the complement of E in $[a, b]$ is dense in $[a, b]$.

2. Let $\{f_n\}$ be a sequence of real-valued measurable functions defined on $[0, 1]$. Show that there exists a sequence of positive real numbers $\{a_n\}$ such that $a_n f_n \to 0$ a.e.

   Hint: Use Lusin's Theorem.

3. Let f be non-negative, bounded and measurable over a set E of finite measure with $\int_E f = 0$. Show that $f = 0$ a.e.

4. Let g be integrable over $[a, b]$, and define $f(x) = \int_a^x g$, $\forall x \in [a, b]$. Show that f is absolutely continuous on $[a, b]$.

5. Define
$$f(x) = \begin{cases} 0 & x \text{ rational} \\ 1/a & x \text{ irrational and } a \text{ is the first non-zero in-} \\ & \text{teger in the decimal expansion of } x. \end{cases}$$
   For example, if $x = 0.002439\ldots$, then $a = 2$.

   (a) Prove that f is measurable.

   (b) Find $\int_0^1 f$ if the integral exists.

6. Suppose that $f \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$.

   (a) Show that $f \in L^p(\mathbb{R})$ for any $1 < p < \infty$.

   (b) Show that $\lim_{p\to\infty} ||f||_p = ||f||_\infty$.

MTH 636: Provide complete solutions to **only** 7 of the 8 problems. * is xtra-credit.

1. Find the set of points z at which $f(z) = |z^4|$ is differentiable. Justify your answer.

2. Assume $R > 0$, $M \geq 0$. Suppose $f(z)$ is analytic on $|z| \leq R$, $f(0) = 0$ and $|f(z)| \leq M$ for $|z| \leq R$. Prove Schwarz's lemma, that is,

$$|f(z)| \leq \frac{M}{R}|z|.$$

3. Show that $8e$, $e = 2.718\ldots$, is an upper bound for $|\int_C z^4 e^z \, dz|$, where C is the straight line contour joining the points $1 + i$ and $1 - i$.

4. If f is analytic for $|z| \leq 1$ then show that

   (a)
$$\frac{1}{\pi} \int_{|z| \leq 1} f(x + iy) dx \, dy = f(0).$$

   *(b) Show also that (Gauss' mean value theorem)
$$\frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) d\theta = f(0).$$

5. If $f(z) = g(z)/h(z)$, where g and h are analytic at $z = \alpha$, but h has a simple zero at $z = \alpha$, then show that the residue of f at α is given by

$$a_{-1} = \frac{g(\alpha)}{h'(\alpha)}.$$

6. Find the Laurent series expansion of

$$f(z) = \frac{2}{(z+3)(z+1)}$$

   in the annulus $1 < |z| < 3$.

7. Use Rouche's Theorem to show that all zeros of the complex-valued polynomial

$$p(z) = z^7 - 5z^3 + 12$$

   lie between the circles $|z| = 1$ and $|z| = 2$.

8. Let $m > 0$. Show that
$$\int_0^{+\infty} \frac{\cos mx}{x^2 + 1} dx = \frac{\pi}{2} e^{-m}.$$