Analysis Qualifying Exam January 9, 2015

MTH 636: Do <u>**five**</u> of the following six questions:

1. Each of the following functions has a singularity at $z = 0$. Determine whether the singularity is an essential singularity, a pole, or a removable singularity. If the singularity is a pole, determine the order of the pole. Justify your answers.

   (a) $z^2 \sin(1/z^2)$

   (b) $\dfrac{2\cos z - 2 + z^2}{z^4}$

   (c) $\dfrac{e^z - 1 + z}{z^5}$

2. Find the Laurent expansion of the function $f(z) = \dfrac{2z+5}{z^2+5z+4}$

   (a) in the open circle $|z| < 1$, and

   (b) in the region $|z| > 4$.

3. Suppose $p(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0$, where $|a_0| + |a_1| + \ldots + |a_{n-1}| < 1$.

   (a) **Prove**: p has n roots in the unit disk $D = \{z : |z| < 1\}$.

   (b) Prove that $p(z) - be^z$ has n roots inside the unit disk if $|b|$ is sufficiently small, and no roots inside the unit disk if b is sufficiently large.

4. Evaluate

$$\frac{1}{2\pi i}\int_{|z|=1}\left(z+\frac{1}{z}\right)^5\left(z^3+\frac{1}{z^3}\right)\frac{dz}{z}.$$

   Using this or otherwise, evaluate

$$\int_0^{2\pi} \cos^5\theta \cos(3\theta)\, d\theta$$

5. Suppose a and b are positive real numbers. Evaluate

$$\int_{-\infty}^{\infty} \frac{\cos(bx)}{x^2+a^2} dx.$$

6. Suppose that f is analytic in a domain that contains the closed unit disk $\{z \in \mathbb{C} : |z| \leq 1\}$. Suppose also that f has a zero of order k at $z = 0$ and that $|f(z)| \leq M$ when $|z| = 1$. **Prove**: If $|z| \leq 1$, then $|f(z)| \leq M|z|^k$.

MTH 632: Do <u>**five**</u> of the following six questions:

1. Let $E \subset \mathbb{R}$ be a measurable set such that $m(E) = 1$. Show that there is a measurable subset $F \subset E$, such that $m(F) = \frac{1}{2}$. (Hint: consider the function $f(x) = m\left((-\infty, x] \cap E\right)$.)

2. Let f_n be a sequence of nonnegative measurable functions on $\mathbb{R}$ that converges pointwise on $\mathbb{R}$ to f. Suppose that $f \in L^1(\mathbb{R})$, and $\lim_{n\to\infty} \int_{\mathbb{R}} f_n = \int_{\mathbb{R}} f$. Then show that for each measurable subset E of $\mathbb{R}$, the limit $\lim_{n\to\infty} \int_E f_n$ exists, and is equal to $\int_E f$.

3. For each of the following, either give an example of the object described, or state that it does not exist. Justify that your example indeed works, and if an example does not exist, justify your statement that such an example does not exist.

   (a) A function in $L^2(\mathbb{R})$ which is not in $L^1(\mathbb{R})$.

   (b) A function in $L^2([0,1])$ which is not in $L^1([0,1])$

   (c) A function f on $[0,1]$ which is differentiable at each point such that the derivative f' is not a measurable function.

4. Let f be an absolutely continuous function on $[a,b]$ and let $\lambda : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function. Show that the composite function $\lambda \circ f$ is absolutely continuous.

5. Let g be a nonnegative and nonzero continuous function on $[0,\infty)$ such that $g(0) = 0$ and $\lim_{t\to\infty} g(t) = 0$. Set $u_n(t) = g(nt)$. Answer the following questions, justifying each of your answers.

   (a) Does the sequence u_n converge uniformly to a limit on $[0,\infty)$ as $n \to \infty$?

   (b) Show that the limit
   $$\lim_{n\to\infty} \int_0^{2\pi} u_n(t)dt$$
   exists, and compute it.

6. Let $E \subset \mathbb{R}$ be a measurable set of finite measure.

   (a) Let $f \geq 0$ be a measurable function on E such that $f \in L^1(E)$ and we have $\int_E f = 0$. What can we say about f? Justify your answer.

(b) Let $g \in L^2(E)$ be such that for each $h \in L^2(E)$ we have

$$\int_E g \cdot h = 2015 \int_E h.$$

What can we say about g? Justify your answer.