Central Michigan University

Department of Mathematics

Analysis Qualifying Examination

January 4, 2017

INSTRUCTIONS

(1) This question paper has 6 problems from Real Analysis (MTH 632) numbered **R1** through **R6** and 6 problems from Complex Analysis numbered **C1** through **C6**.
(2) Do **five** problems from the Real Analysis (MTH 632) part, and another **five** problems from the Complex Analysis (MTH 636) part.
(3) In either section, if you attempt solutions to all six problems, then only the five highest scores will count.
(4) Begin each problem on a separate sheet of paper, and clearly write the problem number and your name before beginning your solution.
(5) Give proper mathematical justification of all your statements.
(6) At the end, turn in the Real Analysis and Complex Analysis solutions in **separately stapled** bunches.

Good Luck!

MTH 636: Do **<u>five</u>** of the following six :

(C1) Let $f \in H(\mathbb{C})$ with $f(z) = u(x,y) + iv(x,y)$. Suppose $au(x,y) + bv(x,y) = c$, $a, b, c \in \mathbb{C} \setminus \{0\}$ with $a \neq \pm bi$. Show that f is constant.

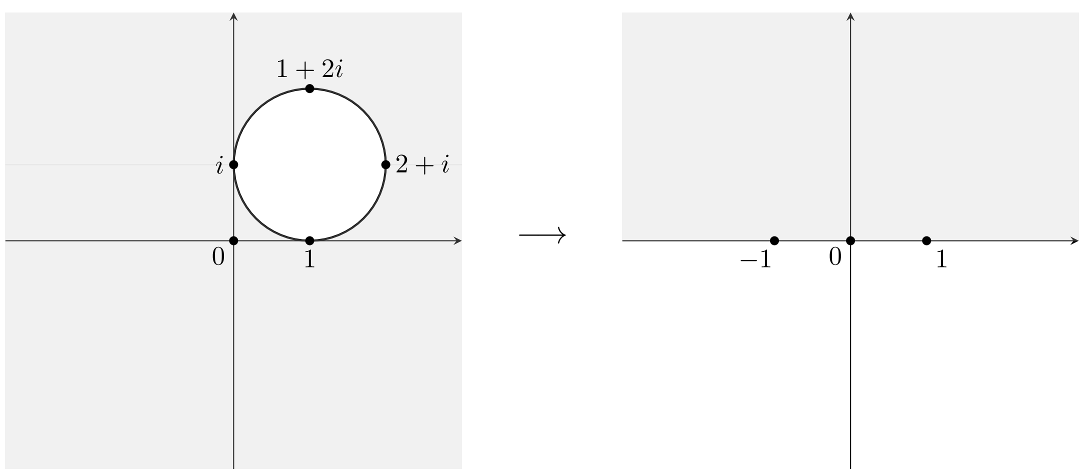
(C2) Evaluate $\displaystyle\int_0^\infty \frac{dx}{1+x^n}$, $n \in \mathbb{N}$, $n \geq 2$. (Hint: Use the boundary of a sector containing only one pole.)

(C3) Let $U \subset \mathbb{C}$ be an open connected set with $0 \in U$. Let $f \in H(U \setminus \{0\})$ and f has 0 as either a pole of order $m \in \mathbb{N}$ or a removable singularity whose removal results in 0 being a zero of order m. Show 0 is a simple pole of $\dfrac{f'}{f}$ and find the residue.

(C4) Let $U \subset \mathbb{C}$ be a bounded open connected set and $f \in H(U)$ and continuous on $\overline{U}$. Suppose $f(z) \neq 0$ for all $z \in \overline{U}$ and $|f(z)| = C$, $z \in \partial U$.
(a) Prove f must be constant on U.
(b) Is the fact that U is bounded necessary?

(C5) Find the number of solutions of the equation $z^8 - 5x^3 + z = 2$ in the region $\{z \in \mathbb{C} : 1 < |z| < 2\}$.

(C6) Find a conformal mapping of the region outside the disk $\{z : |z - 1 - i| < 1\}$ to the upper half plane. Explain why such a mapping must exist. See the figure.



MTH 632: Provide complete solutions to **only** 5 of the 6 problems.

(R1) Define a function f on $[0,1]$ in the following way: for x rational we have $f(x)=0$, and for x irrational we have

$$f(x)=\frac{1}{a}$$

where $a\in\{1,2,\ldots,9\}$ is the first nonzero digit in the decimal expansion of x.

(a) Show that f is measurable.

(b) Compute the Lebesgue integral $\int_{[0,1]} f$

(R2) Let E be a measurable subset of $[0,1]$. Show that there is an $c\in[0,1]$ such that

$$m\left(E\cap[0,c]\right)=\frac{m(E)}{2017}.$$

(R3) Give an example of each of the following objects, or state that they don't exist. In case such an object does not exist, give a short justification why it does not exist.

(a) A function in $L^5([0,1])$ which is not in $L^3([0,1])$.

(b) A sequence of non-negative functions $g_n\in L^1([0,1])$ such that $g_n(x)\to 2x$ for each x as $n\to\infty$, and $\int_{[0,1]} g_n=\frac{1}{2}$.

(c) A continuous function $f\in L^1(\mathbb{R})$ such that for each positive integer n, we have $f(n)=n$.

(R4) Let $f:[0,1]\to\mathbb{R}$ be given by

$$f(x)=\begin{cases} x\cos\left(\frac{\pi}{2x}\right) & \text{for } x\neq 0\\ 0 & \text{for } x=0.\end{cases}$$

Answer the following questions, fully justifying your responses.

(a) Is this function continuous?

(b) Is it uniformly continuous?

(c) Is it of bounded variation?

(d) Is it absolutely continuous?

(R5) For each positive integer n, define a function $f_n:(0,\infty)\to\mathbb{R}$ by setting

$$f_n(x)=\frac{e^{-x}}{1+nx}.$$

(a) Does the sequence of functions f_n converge *pointwise* to a limit function as $n\to\infty$? If yes, what is this limit?

(b) Does the sequence of functions f_n converge *uniformly* to a limit function as $n\to\infty$?

(c) With full justification, compute the limit

$$\lim_{n\to\infty}\int_{(0,\infty)} f_n.$$

(R6) Let $p_1, p_2, \ldots p_n$ be positive numbers such that

$$\sum_{j=1}^{n} \frac{1}{p_j} = 1.$$

For $j = 1, \ldots, n$, let $f_j \in L^{p_j}(\mathbb{R})$. If f is the product $f_1 \ldots f_n$, show that

$$\|f\|_1 \leq \prod_{j=1}^{n} \|f_j\|_{p_j}.$$