Central Michigan University
Department of Mathematics
Analysis Qualifying Exam
August 23, 2018

Instructions

1. The exam will have 6 problems from Complex Analysis (MTH 636), and 6 problems from Real Analysis (MTH 632).
2. You will be required to complete five problems from the Real Analysis (MTH 632) part, and another five problems from the Complex Analysis (MTH 636) part.
3. In either section, if you attempt solutions to all six problems, then only the first five problems will be graded.
4. Begin each problem on a separate sheet of paper, and clearly write the problem number and your name before beginning your solution. Please write your solutions only on one side of the paper.
5. No calculators or other electronic devices are allowed.
6. No questions may be asked during the exam. If a problem appears ambiguous to you, interpret it in a way that makes sense to you, but not in a way that makes it trivial.
7. Give proper mathematical justification of all your statements.
8. Each question is worth 10 points. For questions divided into multiple subquestions, the points for each part are indicated.

MTH 636: Provide complete solutions to **only** 5 of the 6 problems.

(C1) In each point below find all the holomorphic functions with the given properties.

(a) (5 points) f is holomorphic in the unit disc, and for each positive integer n, we have
$$f\left(\frac{1}{n^2+1}\right) = 0.$$

(b) (5 points) f is entire (i.e. holomorphic in the whole complex plane), $f(0) = 1$ and for each $z \in \mathbb{C}$, $z = x + iy$, we have
$$|f(z)| \leq e^{-x}.$$

(C2) (10 points) Let A, B, C, D be the vertices of a rectangle in the plane, and u be the function defined in the closed rectangle whose value at any point P is the product of the distances of P to the four vertices
$$u(P) = \mathrm{dist}(P, A) \cdot \mathrm{dist}(P, B) \cdot \mathrm{dist}(P, C) \cdot \mathrm{dist}(P, D).$$

Show that the maximum value of u is attained on the boundary of the rectangle.

(C3) (a) (4 points) If C is the circle $|z| = 3$ traversed once, show that
$$\left| \int_C \frac{dz}{z^2 - i} \right| \leq \frac{3\pi}{4}.$$

(b) (6 points) Prove that all the roots of $z^4 - z^3 + 7 = 0$ lie in the annulus $1 < |z| < 2$.

(C4) Find the Laurent series representation of the function
$$f(z) = \frac{1}{(z-1)^2(z+1)^2}$$

in the following annuli

(a) (3 points) $\{|z| < 1\}$

(b) (3 points) $\{|z| > 1\}$

(c) (4 points) $\{|z + 1| > 2\}$.

(C5) (a) (6 points) Suppose that f and g are holomorphic near z_0 and g has a simple zero at z_0 and $f(z_0) \neq 0$. Then, show that $\dfrac{f}{g}$ has a simple pole at z_0 and

$$\operatorname{Res}\left(\frac{f}{g}, z_0\right) = \frac{f(z_0)}{g'(z_0)}$$

(b) (4 points) Compute the residue of $F(z) = \dfrac{e^z}{\sin(z)\cos(z)}$ at $z_0 = \pi$.

(C6) (10 points) Show that $\displaystyle\int_0^{2\pi} \frac{\sin^2(\theta)}{5 + 4\cos\theta} d\theta = \frac{\pi}{4}$ using Residue theorem.

MTH 632: Provide complete solutions to **only** 5 of the 6 problems.

(R1) (10 points) Let the function f be defined on a measurable set E. Show that f is measurable if and only if for each Borel set A, $f^{-1}(A)$ is measurable.

(R2) (a) (5 points) Does the bounded convergence theorem hold for the Riemann integral? Justify your answer.

(b) (5 points) Let $\{f_n\}$ be a sequence of nonnegative measurable functions on E that converges pointwise on E to f. Suppose $f_n \leq f$ on E for each n. Show that $\lim_{n\to\infty} \int_E f_n = \int_E f$.

(R3) (10 points) Let $g \in L^1(\mathbb{R})$, and define a function f on $\mathbb{R}$ by setting:

$$f(x) = \int_{\mathbb{R}} \frac{g(y)}{1 + x^2y^2} dy$$

where the integral on the right is a Lebesgue integral. Show that f is a continuous function on $\mathbb{R}$.

(R4) (10 points) Let f be integrable over $(-\infty\ , \infty)$. Show that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x) \cos nx \ dx = 0.$$

(R5) (10 points) Let f be an absolutely continuous function on $[0, 1]$ and let $\lambda : \mathbb{R} \to \mathbb{R}$ satisfy

$$|\lambda(x) - \lambda(y)| \leq 2018\, |x - y|\, .$$

Using the definition of absolute continuity of a function show that the composite function $\lambda \circ f$ is also absolutely continuous.

(R6) For each of the following, either give an example of the object described, or state that it does not exist. Justify that your example indeed works, and if an example does not exist, justify your statement that such an example does not exist.

(a) (3 points) A function in $L^1(\mathbb{R})$ which is not in $L^2(\mathbb{R})$.

(b) (3 points) A function in $L^2([0, 1])$ which is not in $L^1([0, 1])$

(c) (4 points) A function f on $\mathbb{R}$ which is differentiable at each point such that the derivative f' is not a measurable function.