Central Michigan University
Department of Mathematics
Analysis Qualifying Exam
January 4, 2018

Instructions

1. The exam will have 6 problems from Complex Analysis (MTH 636), and 6 problems from Real Analysis (MTH 632).
2. You will be required to complete five problems from the Real Analysis (MTH 632) part, and another five problems from the Complex Analysis (MTH 636) part.
3. In either section, if you attempt solutions to all six problems, then only the first five problems will be graded.
4. Begin each problem on a separate sheet of paper, and clearly write the problem number and your name before beginning your solution.
5. No calculators or other electronic devices are allowed.
6. No questions may be asked during the exam. If a problem appears ambiguous to you, interpret it in a way that makes sense to you, but not in a way that makes it trivial.
7. Give proper mathematical justification of all your statements.
8. Each question is worth 10 points. For questions divided into multiple subquestions, the points for each part are indicated.

MTH 636: Provide complete solutions to **only** 5 of the 6 problems.

(C1) [10 points] Show that if f is holomorphic in $|z| \leq 2$, there must be some positive integer n such that $f(1/n) \neq \dfrac{1}{n+1}$

(C2) (a) [5 points] Let f be entire and suppose that $\mathrm{Re} f(z) \leq \mathrm{Im} f(z)$ for all z in $\mathbb{C}$. Prove that f must be a constant function.

(b) [5 points] Suppose that f is an entire function, n is a positive integer, and $|f(z)| \leq (1+|z|)^n$ for all $z \in \mathbb{C}$. Show that f is a polynomial.

(C3) [10 points] Suppose f and g are both holomorphic in a domain D which is also compact. Show that $|f(z)|+|g(z)|$ takes its maximum value on the boundary ∂D.

(C4) [10 points] Find the Laurent series of the form $\sum\limits_{n=-\infty}^{\infty} c_n z^n$ for $f(z) = \dfrac{3}{(2z-1)(z+5)}$ that converges in an annulus containing the point $z = i$, and state precisely where this Laurent series converges.

(C5) [10 points] Let n be a positive integer and let a be a real number satisfying $a > e$. Show that the equation $e^z = az^n$ has n solutions inside the unit circle.

(C6) (a) [6 points] Evaluate p.v. $\int\limits_{-\infty}^{\infty} \dfrac{e^{ix}}{x}\, dx$. Justify all the steps involved in finding your answer.

(b) [4 points] Find $\int\limits_{0}^{\infty} \dfrac{\sin x}{x}\, dx$ using your answer to part 6(a).

MTH 632: Provide complete solutions to **only** 5 of the 6 problems.

(R1) Let $f : [0,1] \to \mathbb{R}$ the function defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ x^{-\frac{1}{3}} & \text{if } x \text{ is irrational.} \end{cases}$$

(a) (3 points) Show that f is measurable.

(b) (3 points) With full justification, find the value of the Lebesgue integral

$$\int_{[0,1]} f.$$

(c) (4 points) Find the values of p, with $1 \leq p \leq \infty$, for which $f \in L^p([0,1])$.

(R2) Let $\{f_n\}$ be a sequence of non-negative measurable functions on the real line. Suppose that we have as $n \to \infty$,

$$f_n(x) \to \frac{1}{1+x^2} \quad \text{for almost every } x, \text{ and } \int_{\mathbb{R}} f_n \to \pi.$$

With full justification, find the value of

$$\lim_{n\to\infty} \int_{[0,1]} f_n.$$

(R3) Let $\{\phi_k\}_{k=1}^{\infty}$ be an orthonormal sequence in the Hilbert space $L^2([0,1])$.

(a) (4 points) Suppose that each ϕ_k is real valued and $\phi_1 > 0$ a.e. Show that for each $k \geq 2$, the set

$$\{x \in [0,1] : \phi_k(x) < 0\}$$

has positive measure.

(b) (6 points) Show that partial sums of the series

$$\sum_{k=1}^{\infty} \frac{k^2}{k^3+1} \phi_k$$

converge in the L^2-norm to a function in $L^2([0,1])$.

(R4) Let $\{f_n\}$ be a sequence of measurable functions on $[0,1]$ and let f be a measurable function on $[0,1]$. For each of the following statements, state whether the statement is **true** or **false**. For each true statement, provide a short proof. For each false statement, provide a counterexample to show that the statement is false. You don't need to provide justification that the counterexample works.

(a) (3 points) If $f_n \to f$ uniformly and each f_n as well as f is in $L^2([0,1])$, then $f_n \to f$ in $L^2([0,1])$.

(b) (4 points) If $f_n \to f$ a.e. and each f_n as well as f is in $L^1([0,1])$, then $f_n \to f$ in $L^1([0,1])$.

(c) (3 points) If $f_n \to f$ in $L^3([0,1])$, then $f_n \to f$ in $L^2([0,1])$.

(R5) Let $f \in L^1([0,1])$, and for each positive integer n, let

$$E_n = \{x \in [0,1] \,|\, |f(x)| > n\}.$$

Let $m(E)$ denote the Lebesgue measure of a set E.

(a) (5 points) Show that

$$\lim_{n\to\infty} m(E_n) = 0.$$

(b) (5 points) Show that

$$\lim_{n\to\infty} \int_{E_n} f = 0.$$

(R6) For each of the following, either give an example of the object described, or state that such an object does not exist. If according to you such an object does not exist, justify your answer in at most three short sentences.

(a) (3 points) A nonmeasurable function $f : \mathbb{R} \to \mathbb{R}$, such that the function $g : \mathbb{R} \to \mathbb{R}$, defined by

$$g(x) = f(|x|)$$

is measurable.

(b) (3 points) A sequence of non-negative continuous functions $f_n : \mathbb{R} \to \mathbb{R}$ such that $f_n \to 0$ uniformly on $\mathbb{R}$, but for each n, we have

$$\int_{\mathbb{R}} f_n = 1$$

(c) (4 points) A measurable function $f : [0,1] \to \mathbb{R}$ such that

$$\int_0^1 (f(x))^4\,dx = 125 \quad \text{and} \quad \int_0^1 x^3 f(x)dx = 2,$$

where the integrals are Lebesgue integrals.