Analysis Qualifying Exam: September 4, 2009

MTH 632: Provide complete solutions to 5 of the 6 questions.

Notation: $\mathbb{Q}$ denotes the set of rational numbers and $\mathbb{R}$ denotes the set of real numbers.

1. Assume $A, B, G \subset \mathbb{R}$ and m^* is Lebesgue outer measure on $\mathbb{R}$. Here $\widetilde{G}$ denotes the complement of G.

   (a) Suppose G is measurable and $A \subset G$ and let B be such that $B \cap G = \emptyset$. Show $m^*(A \cup B) = m^*(A) + m^*(B)$.

   (b) Suppose A and B are such that $\operatorname{dist}(A, B) = \inf\{|x - y| : x \in A,\ y \in B\} > 0$. Show $m^*(A \cup B) = m^*(A) + m^*(B)$.

2. Let $f(x) = \begin{cases} x(1-x) & ; \quad x \in [0,1] \backslash \mathbb{Q} \\ 1 & ; \quad x \in [0,1] \cap \mathbb{Q} \end{cases}$. Find $\displaystyle\int_{[0,1]} f\, dm$. Is f Riemann integrable? Explain your answer.

3. Let $f \in L^1(\mathbb{R})$. Show there is a sequence $\langle x_n \rangle \subset \mathbb{R}$ with $\lim_{n\to\infty} x_n = \infty$ such that $\lim_{n\to\infty} x_n f(x_n) = 0$.

4. Let g be a function defined on $\mathbb{R}$ such that there is a constant $\lambda > 0$ such that

   $$|g(x) - g(y)| \leq \lambda |x - y|, \quad \forall\, x, y \in \mathbb{R},$$

   i.e. g satisfies a Lipschitz condition on $\mathbb{R}$ and is hence continuous. Let $f \in L^1([a, b])$. Show $g \circ f$, the composition of f and g is Lebesgue integrable on $[a, b]$.

5. Let $\langle f_n \rangle$ be a sequence of integrable functions defined on a measurable set $E \subset \mathbb{R}$. The sequence $\langle f_n \rangle$ is said to be **equi-integrable** on E if $\forall\ \epsilon > 0, \exists\ \delta > 0$ such that $\forall$ measurable sets $A \subset E$ with $m(A) < \delta$ we have $\displaystyle\int_A |f_n|\, dm < \epsilon,\ \forall\ n$. Suppose $\langle f_n \rangle$ is a convergent sequence, say $f_n \to f$, of equi-integrable functions on a measurable set E, $m(E) < \infty$. Then $\displaystyle\lim_{n\to\infty} \int_E f_n\, dm = \int_E f\, dm$.

6. Let $\langle f_n \rangle$ be a sequence of nonnegative measurable functions on a set E such that $\displaystyle\lim_{n\to\infty} \int_E f_n\, dm = 0$. Show $\langle f_n \rangle$ converges to zero in measure. Show convergence in measure cannot be replaced with convergence almost everywhere.

MTH 636: Provide complete solutions to 6 of the 7 questions

1. Let $f(z) = \begin{cases} \dfrac{x^{\frac{4}{3}}y^{\frac{5}{3}} + ix^{\frac{5}{3}}y^{\frac{4}{3}}}{x^2+y^2} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$

   Show that the Cauchy-Riemann equations hold at $z = 0$, but f is not differentiable at $z = 0$.

2. If $f(z) = u(x,y) + iv(x,y)$ is *entire* such that $au + bv \geq c$ for some real numbers a, b, and c, must f be constant? Prove your answer.

3. Let g be a continuous complex-valued function of a real variable on $[0,2]$, and for each complex number z define

$$F(z) := \int_0^2 e^{zt} g(t)dt.$$

   Prove that F is entire, and find its power series around the origin.

4. Find the Laurent series for the function

$$f(z) = \frac{z}{(z+1)(z-2)}$$

   in each of the following domains:

   (a) $|z| < 1$ (b) $1 < |z| < 2$ (c) $2 < |z|$

5. Does there exist a function $f(z)$ analytic in $|z| < 1$ and satisfying $f(\frac{1}{2}) = \frac{1}{2}$, $f(\frac{1}{3}) = \frac{1}{2}$, $f(\frac{1}{4}) = \frac{1}{4}$, $f(\frac{1}{5}) = \frac{1}{4}$, $\ldots$, $f(\frac{1}{2n}) = \frac{1}{2n}$, $f(\frac{1}{2n+1}) = \frac{1}{2n}$, $\ldots$? Justify your answer.

6. Show that all roots of $z^5 - 3z^2 - 1 = 0$ lie inside the circle $|z| = 2^{\frac{2}{3}}$ and two of its roots lie inside the circle $|z| = \frac{3}{4}$.

7. Prove that $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2+9)^2} dx = \frac{\pi}{6}$.