CENTRAL MICHIGAN UNIVERSITY
DEPARTMENT OF MATHEMATICS
ANALYSIS QUALIFYING EXAM
AUGUST 28, 2020

INSTRUCTIONS

1. The exam will have 6 problems from Complex Analysis (MTH 636), and 6 problems from Real Analysis (MTH 632).
2. You will be required to complete five problems from the Real Analysis (MTH 632) part, and another five problems from the Complex Analysis (MTH 636) part.
3. In either section, if you attempt solutions to all six problems, then only the first five problems will be graded.
4. Begin each problem on a separate sheet of paper, and clearly write the problem number and your name before beginning your solution.
5. No calculators or other electronic devices are allowed.
6. No questions may be asked during the exam. If a problem appears ambiguous to you, interpret it in a way that makes sense to you, but not in a way that makes it trivial.
7. Give proper mathematical justification of all your statements.
8. Each question is worth 10 points. For questions divided into multiple subquestions, the points for each part are indicated.

MTH 636: Provide complete solutions to **only** 5 of the 6 problems.

(C1) Construct a holomorphic function $f = u + iv$ with real part $u(x, y) = x^2 + y^2$, or prove that such a function does not exist.

(C2) Let g be an entire function, and suppose that there exist real numbers A and B such that $|g(z)| \leq A + B|z|^{3/2}$ for all $z \in \mathbb{C}$. Show that g is a linear polynomial.

(C3) Use the Residue Theorem to evaluate the integral

$$\int_0^\infty \frac{dx}{x^4 + x^2 + 1}.$$

Make sure to express your answer as a real number.

(C4) Find the number of zeroes, counted with multiplicities, of the function $h(z) = z^6 - 5z^4 + 3z^2 - 1$ in the disk $\{z : |z| < 1\}$.

(C5) Let $w \in \mathbb{C}$, let $D'(w, R)$ be a punctured disk centered at w of radius $R > 0$, and let $\{z_n\}$ be a sequence of complex numbers in $D'(w, R)$ such that $z_n \to w$. Denote $D = D'(w, R) \setminus \cup_{n=1}^{\infty} \{z_n\}$, and suppose that f is holomorphic on D and has poles at the points z_n (note that w is *not* an isolated singularity of f, since f is not defined in an open neighborhood of w). Show that $f(D)$ is dense in $\mathbb{C}$. (Hint: Assume that there is $c \in \mathbb{C}$ and $\delta > 0$ such that $|f(z) - c| > \delta$ for all $z \in D$, and consider $g(z) = 1/(f(z) - c)$).

(C6) Let Λ be the period lattice generated by 1 and i, and let

$$P = \{z : 0 \leq \text{Re } z < 1,\ 0 \leq \text{Im } z < 1\}$$

be the fundamental parallelogram. For each of the cases below, either give an explicit example (in terms of σ, ζ, and/or $\wp$) of an elliptic function with period lattice Λ that satisfies the given properties, or prove that such a function does not exist.

(a) An elliptic function of degree 3 whose poles in P are all located in the square $\{z : 0 < \text{Re } z < 1/5,\ 0 < \text{Im } z < 1/5\}$, and whose zeroes in P are all located on the real axis.

(b) An elliptic function having a pole of order 4 at $z = 0$, no other poles in P, and such that its integral along the circle $|z| = 3/2$, in the counterclockwise direction, is equal to 9.

(c) An elliptic function of degree 4 whose derivative has degree 7 (the degree of an elliptic function is the number of poles in P, counted with multiplicity).

MTH 632: Provide complete solutions to **only** 5 of the 6 problems.

(R1) (5points) If (f_n) is a sequence of measurable functions on $\mathbb{R}$ that converges uniformly to f, is it true that $\lim_{n\to\infty} \int |f_n - f| = 0$? If the statement is true, then write a proof, and if not, provide a detailed counterexample.

(5points) If (f_n) is a sequence of measurable functions on $[0,1]$ that converges in mean to a function f, that is, $\lim_{n\to\infty} \int |f_n - f| = 0$, is it true that (f_n) converges uniformly to f? If the statement is true, then write a proof, and if not, provide a detailed counterexample.

(R2) Construct an example of a continuous function f on $E = [0,\infty)$ that satisfies the following:

(i). For each $\alpha > 0$ there is a (Lebesgue) measurable set $E_\alpha \subset E$ such that the measure of E_α is positive, and $f > \alpha$ on E_α.

(ii). $f \in L^1(E)$.

(R3) Let $h \in L^1([0,1])$ and consider f defined for each $y \in \mathbb{R}$ by

$$f(y) = \int_{[0,1]} h(x) \cos\left(x^3 y\right) dx,$$

as a Lebesgue integral with respect to the variable x for each fixed $y \in \mathbb{R}$. Prove that f is continuous on $\mathbb{R}$.

(R4) Suppose $1 < p < q < \infty$.

(a) Give a detailed description of a function f that belongs to $L^p((0,\infty))$ but not to $L^q((0,\infty))$.

(b) Give a detailed description of a function f that belongs to $L^q((0,\infty))$ but not to $L^p((0,\infty))$.

(R5) Suppose a function f defined on $\mathbb{R}$ satisfies that for every rational number q, the set $f^{-1}((-\infty, q))$ is measurable. Must f be a measurable function? Either prove it or explain a counterexample.

(R6) Recall that a property holds *a.e.* (almost everywhere) if the set of points where it fails to hold has measure 0.

(a) Give a detailed description of a function f on $[0,1]$ that is not continuous at any point, but is equal to a continuous function g a.e.

(b) Give a detailed description of a function f on $[0, 1]$ that is continuous a.e. but not equal to any continuous function g.