Central Michigan University
Department of Mathematics
Analysis Qualifying Exam
January 07, 2021

Instructions

1. The exam will have 6 problems from Complex Analysis (MTH 636), and 6 problems from Real Analysis (MTH 632).

2. You will be required to complete five problems from the Real Analysis (MTH 632) part, and another five problems from the Complex Analysis (MTH 636) part.

3. In either section, if you attempt solutions to all six problems, then only the first five problems will be graded.

4. Begin each problem on a separate sheet of paper, and clearly write the problem number and your name before beginning your solution.

5. No calculators or other electronic devices are allowed.

6. No questions may be asked during the exam. If a problem appears ambiguous to you, interpret it in a way that makes sense to you, but not in a way that makes it trivial.

7. Give proper mathematical justification of all your statements.

8. Each question is worth 10 points. For questions divided into multiple subquestions, the points for each part are indicated.

MTH 636: Provide complete solutions to **only** 5 of the 6 problems.

(C1) Find all real numbers a for which there exists an entire function f such that $u(x,y) = x^2 + ay^2$, where u is the real part of f. For each such a that you find, construct an example of such a function.

(C2) State and prove the Fundamental Theorem of Algebra.

(C3) Let f_n be a sequence of entire functions that converges uniformly to a bounded function f (the convergence is uniform over $\mathbb{C}$, and not just compact subsets of $\mathbb{C}$). Prove that there is an integer N such that the function f_n is a constant function for every $n \geq N$.

(C4) Let $a > 0$ be a real number. Use the Residue Theorem to evaluate the integral

$$\int_0^\infty \frac{x^2\,dx}{a^4 + x^4}.$$

Express your answer as a real number.

(C5) Determine the number of zeroes (counted with multiplicities) of the polynomial

$$f(z) = 3 - 7z + 9z^4 + z^7$$

that are located *outside* the disk $\{z : |z| \leq 2\}$. State the theorems that you are using.

(C6) Let $0 < \alpha < 1$ be a real number, let $f \in H(D'(z_0, r))$ be a holomorphic function defined on the punctured disk

$$D'(z_0, r) = \{z : 0 < |z - z_0| < r\},$$

and suppose that f satisfies

$$|f(z)| \leq c|z - z_0|^{-\alpha}$$

for some real number $c > 0$ and for all $z \in D'(z_0, r)$. Show that f has a removable singularity at z_0.

MTH 632: Provide complete solutions to **only** 5 of the 6 problems.

(R1) Suppose $g \in L^1([0,1])$ and consider F defined for each $t \in \mathbb{R}$ by

$$F(t) = \int_{[0,1]} g(x) \sin^3 \left(x^2 t\right) dx,$$

as a Lebesgue integral with respect to the variable x for each fixed $t \in \mathbb{R}$. Prove that F is continuous on $\mathbb{R}$.

(R2) Let E be a nonmeasurable subset of $[0,1]$, and define f on $\mathbb{R}$ by

$$f(x) = \begin{cases} e^x, & \text{if } x \in E, \\ -e^x, & \text{if } x \notin E. \end{cases}$$

(a) (5points) Prove that for every $a \in \mathbb{R}$, $f^{-1}(\{a\})$ is measurable.

(b) (5points) Is f a measurable function? Justify your response.

(R3) Let $E = (0, \infty)$.

(a) Give a detailed description of a function f that is positive on E and belongs to $L^\pi(E)$ but not to $L^{15}(E)$.

(b) Give a detailed description of a function g that is positive on E and belongs to $L^{15}(E)$ but not to $L^\pi(E)$.

(R4) Recall that a sequence (f_n) of measurable functions on a measurable set E *converges in measure on* E to a measurable function f if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} m(\{x \in E : |f_n(x) - f(x)| > \varepsilon\}) = 0.$$

(a) (5points) Prove that if (f_n) converges uniformly to f on $\mathbb{R}$, then (f_n) converges in measure to f on $\mathbb{R}$.

(b) (5points) Does part (a) still hold if "converges uniformly on $\mathbb{R}$" is replaced by "converges pointwise almost everywhere on $\mathbb{R}$"? Justify your response.

(R5) Suppose E is measurable and of positive, finite measure. Prove that if (f_n) is a sequence in $L^\infty(E)$ and converges to f in $L^\infty(E)$, then for each $1 < p < \infty$, every f_n belongs to $L^p(E)$, and (f_n) converges to f in $L^p(E)$.

(R6) If a function is continuous on $\mathbb{R}$, must there be an open interval on which the function is monotone? Justify your response.