Central Michigan University
Department of Mathematics
Analysis Qualifying Exam
August 26, 2021

Instructions

1. The exam will have 6 problems from Complex Analysis (MTH 636), and 6 problems from Real Analysis (MTH 632).
2. You will be required to complete five problems from the Real Analysis (MTH 632) part, and another five problems from the Complex Analysis (MTH 636) part.
3. In either section, if you attempt solutions to all six problems, then only the first five problems will be graded.
4. Begin each problem on a separate sheet of paper, and clearly write the problem number and your name before beginning your solution.
5. No calculators or other electronic devices are allowed.
6. No questions may be asked during the exam. If a problem appears ambiguous to you, interpret it in a way that makes sense to you, but not in a way that makes it trivial.
7. Give proper mathematical justification of all your statements.
8. Each question is worth 10 points. For questions divided into multiple subquestions, the points for each part are indicated.

MTH 636: Provide complete solutions to **only** 5 of the 6 problems.

(C1) Determine all values of $\alpha \in \mathbb{C}$ for which there exists a holomorphic function f on an open set V of $\mathbb{C}$ having imaginary part $v(x, y) = 3x^2y - 2\alpha y^3 + 2xy$. Justify your answer.

(C2) Determine the number of zeros (counted with multiplicities) of the polynomial

$$f(z) = 3 - 5z + 7z^4 + z^7$$

that are located in the open disk $\{z : |z| < 2\}$. State the theorems that you are using.

(C3) a) (3 points) State Morera's Theorem.

b) (7 points) Let γ be a piecewise smooth closed curve in $\mathbb{C}$, and let $g \colon \gamma^* \to \mathbb{C}$ be a continuous function, where γ^* is the image of γ. Further, let f be an entire function, and suppose that

$$F(z) = \int_\gamma f(z - w)g(w)\, dw$$

is continuous at all $z \in \mathbb{C}$. Prove that F is entire.

(C4) Let f be an entire function with infinitely many distinct zeros. Prove that the set of these zeros is not bounded.

(C5) Let $a > 2$ be a real number. Evaluate the integral

$$\int_\gamma \frac{dz}{(z - a)(z^3 - 1)},$$

where $\gamma = \{z : |z| = 2\}$ is the circle of radius two centered at the origin, traversed counterclockwise. Simplify your answer.

(C6) Let $f(z)$ be an entire function, and suppose that there exist positive real numbers A, B, and k such that

$$|f(z)| \leq A + B|z|^k$$

for all $z \in \mathbb{C}$. Show that f is a polynomial.

MTH 632: Provide complete solutions to **only** 5 of the 6 problems.

(R1) (a) (5 points) Give an example of a sequence of integrable functions (f_n) on $\mathbb{R}$ such that (f_n) converges pointwise almost everywhere to a function f, but does not converge in mean to f; that is, $\lim_{n\to\infty} \int |f_n - f|$ is not zero.

(b) (5 points) Recall that a sequence (f_n) of measurable functions on a measurable set E *converges in measure on* E to a measurable function f if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} m(\{x \in E : |f_n(x) - f(x)| > \varepsilon\}) = 0.$$

Give an example of a sequence of integrable functions (f_n) on $\mathbb{R}$ such that (f_n) converges pointwise almost everywhere to a function f, but does not converge in measure to f.

(R2) Let $(X, ||\cdot||)$ be a normed space. For two subsets A and D of X with $D \subseteq A$, the set D is *dense in* A if given any $x \in A$ and any $\varepsilon > 0$ there exists $y \in D$ such that $||x - y|| < \varepsilon$.

(a) (5 points) Prove that D is dense in A if and only if, given any $x \in A$, there is a sequence (y_n) from D such that $y_n \to x$ in the norm of X.

(b) (5 points) Assume that $(X, ||\cdot||)$ is a Banach space. Suppose F is a subspace of X that is dense in X. Prove that if F (equipped with the norm $||\cdot||$ induced from X) is also a Banach space, then $F = X$.

(R3) Let $E = [0, \infty)$.

(a) (5 points) Give an example of a function f on E such that f is nonzero on a set of positive measure and for every $1 \le p \le \infty$, $f \in L_p(E)$, but there is at least one point in E where f is discontinuous.

(b) (5 points) Give an example of a continuous function f on E such that for every $1 \le p \le \infty$, $f \notin L_p(E)$.

(R4) Assume $[a, b]$ is a nontrivial interval. Suppose φ is a positive, continuous function on $[a, b]$. Denote by φ_n the function $\varphi^{1/n}$ and suppose that (φ_n) converges pointwise on $[a, b]$ to the constant function $\underline{1}$. Let f and g be integrable functions on $[a, b]$ such that for every $n \in \mathbb{N}$,

$$\int_{[a,b]} (f - g)\varphi^{1/n} = 0.$$

Prove that $f = g$ (a.e.) on $[a, b]$.

(R5) Consider A a non-measurable subset of $(0,1)$ Define a function f by:

$$f(x) = \begin{cases} -x^2, & \text{if } x \in (-\infty, -2), \\ -4, & \text{if } x \in [-2, 0], \\ x^2, & \text{if } x \in A. \end{cases}$$

(a) Prove that for every $a \in \mathbb{R}$, $f^{-1}(\{a\})$ is measurable.

(b) Is f a measurable function? Justify your response.

(R6) Assume f is integrable over the measurable set E. Suppose (E_n) is an increasing sequence of measurable subsets of E. Prove that

$$\int_{\bigcup_{n=1}^{\infty} E_n} f = \lim_{n\to\infty} \int_{E_n} f.$$