CENTRAL MICHIGAN UNIVERSITY
DEPARTMENT OF MATHEMATICS
ANALYSIS QUALIFYING EXAM
AUGUST 26, 2022

Each question of this exam is worth 10 points. For questions divided into multiple sub-questions, the points for each part are indicated.

1. Instructions

(1) The exam will have 6 problems from Complex Analysis (MTH 636), and 6 problems from Real Analysis (MTH 632).

(2) You will be required to complete 5 problems from the Complex Analysis (MTH 636) part, and another 5 problems from the Real Analysis (MTH 632) part.

(3) In either section, if you attempt solutions to all 6 problems, then only the first 5 problems will be graded.

(4) Begin each problem on a separate sheet of paper, and clearly write the problem number and your name before beginning your solution.

(5) No calculators or other electronic devices are allowed.

(6) No questions may be asked during the exam. If a problem appears ambiguous to you, interpret it in a way that makes sense to you, but not in a way that makes it trivial.

(7) Give proper mathematical justification for all your statements.

MTH 636: Provide complete solutions to **only** 5 of the 6 problems.

(1) Construct a holomorphic function $f = u + iv$ on the complex plane with imaginary part $v(x, y) = x - x^2 + y^2$, or show that such a function does not exist.

(2) Determine the number of zeroes of the function $f(z) = 2 + 7z^2 + 3z^4 + z^7$ in the annulus

$$A(0, 1, 3) = \{z : 1 < |z| < 3\}.$$

(3) Let $a > 0$ be a real number. Evaluate the integral

$$\int_0^\infty \frac{x^2\,dx}{(a^2 + x^2)^2}.$$

Express your answer as a real number.

(4) Let f be a holomorphic function on the punctured disk

$$D'(0, 2) = \{z : 0 < |z| < 2\}.$$

Suppose that there exist real constants C and $\alpha < 2$ such that

$$|f(z)| \leq \frac{C}{|z|^\alpha}$$

for all $z \in D'(0, 2)$. Finally, suppose that

$$\int_\gamma f(z)dz = 0,$$

where $\gamma = \{z : |z| = 1\}$ is the unit circle, oriented clockwise. Prove that f has a removable singularity at $z = 0$.

(5) Suppose f and g are entire functions, and suppose that $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{C}$. Show that $f(z) = Cg(z)$ for some complex number C.

(6) Let $f(z)$ be a polynomial of degree at least 1. Show that the function $e^{f(1/z)}$ has an essential singularity at $z = 0$.

MTH 632: Provide complete solutions to **only** 5 of the 6 problems.

(1) Let $E \subseteq \mathbb{R}$ be a set with finite Lebesgue outer measure $m^*(E)$. Prove using the definitions of the Lebesgue outer measure and of measurability that E is measurable *if and only if* there exists a G_δ-set G such that
- $E \subseteq G$,
- $m^*(E) = m^*(G)$, and
- $m^*(G \setminus E) = 0$.

(2) Let $E \subseteq \mathbb{R}$ be a subset of finite measure, and $(f_n \colon E \to \mathbb{R})$ a sequence of measurable functions that converges pointwise a.e. to $f \colon E \to \mathbb{R}$. Show that

$$\lim_{n\to\infty} \int_E \frac{|f_n|}{1+|f_n|} = \int_E \frac{|f|}{1+|f|}.$$

(3) Recall that for any $A \subseteq \mathbb{R}$, we define $\chi_A(x)$ to be 1 if $x \in A$ and 0 otherwise. Each of the following sequences of measurable functions $(f_n \colon E \to \mathbb{R})_{n=1}^\infty$ converges to 0 pointwise a.e.:

(a) $E = [0, 1/2]$ and $f_n(x) = x^n$ for $x \in [0, 1/2]$,

(b) $E = \mathbb{R}$ and $f_n = \chi_{[n,n+1]}$ for each n,

(c) $E = [-1, 1]$ and $f_n = n\chi_{[-1/n,1/n]}$.

Choose *two* of the sequences. For each chosen sequence, determine

(i) whether it converges uniformly to 0,

(ii) whether it converges in measure to 0, and

(iii) whether it converges to 0 in $L^1(E)$.

Justify your solutions. (If you provide solutions for three sequences, only the first two will be graded.)

(4) Suppose $h \in L^1([0,1])$ and define f for each $t \in \mathbb{R}$ by

$$f(t) = \int_{[0,1]} h(x) \sin^3\left(tx^5\right) dx.$$

Here, for each fixed $t \in \mathbb{R}$, the integral is a Lebesgue integral with respect to the variable x. Prove that f is continuous on $\mathbb{R}$.

(5) Construct an example of a function f on $E = [0, \infty)$ that satisfies the following:

(i) The function f is continuous on E.

(ii) For each positive integer n, $f(n) \geq n^2$.

(iii) $f \in L^1(E)$.

(6) Suppose $1 \leq p_1 < p_2 < \infty$.

(a) Let $E \subset \mathbb{R}$ have finite measure. Prove $L^{p_2}(E) \subseteq L^{p_1}(E)$.

(b) If $E = [0, 1]$, prove that $L^{p_1}(E) \neq L^{p_2}(E)$.