# CENTRAL MICHIGAN UNIVERSITY
# DEPARTMENT OF MATHEMATICS
# ANALYSIS QUALIFYING EXAM
# AUGUST 24, 2023

Each question of this exam is worth 10 points. For questions divided into multiple subquestions, the points for each part are indicated.

## 1. Instructions

(1) The exam will have 6 problems from Complex Analysis (MTH 636), and 6 problems from Real Analysis (MTH 632).

(2) You will be required to complete 5 problems from the Complex Analysis (MTH 636) part, and another 5 problems from the Real Analysis (MTH 632) part.

(3) In either section, if you attempt solutions to all 6 problems, then only the first 5 problems will be graded.

(4) Begin each problem on a separate sheet of paper, and clearly write the problem number and your name before beginning your solution.

(5) No calculators or other electronic devices are allowed.

(6) No questions may be asked during the exam. If a problem appears ambiguous to you, interpret it in a way that makes sense to you, but not in a way that makes it trivial.

(7) Give proper mathematical justification for all your statements.

(8) Throughout the exam, if a question has multiple parts, you may assume any previous part as a true statement in order to answer the consecutive parts whenever needed.

**MTH 636:** Provide complete solutions to **only** 5 of the 6 problems.

(1) Let $f$ be a complex-valued continuously differentiable function on an open set $U \subset \mathbb{C}$. Suppose that $f(z) \neq 0$ for each $z \in U$ and that $f^2$ is holomorphic on $U$. Prove that $f$ is holomorphic on $U$.

(2) Find all holomorphic functions $f\colon \mathbb{C} \to \mathbb{C}$ such that $f(0) = 1$ and $\operatorname{Re} f(z) \leq 1$ for each $z \in \mathbb{C}$.

(3) Let $\mathbb{D}$ be the unit disc $\{|z| < 1\}$. Find all holomorphic functions $f\colon \mathbb{D} \to \mathbb{C}$ such that for each integer $n \geq 2$ we have $f\left(\frac{1}{n}\right) = \frac{1}{n^2}$.

(4) Let $\gamma$ denote the unit circle in the complex plane $\{|z| = 1\}$, oriented counterclockwise. Compute the following line integrals:

(a) **(4 points)** $\displaystyle\int_\gamma \frac{\sin(\pi z)}{3z-1}dz$,

(b) **(2 points)** $\displaystyle\int_\gamma \overline{z}dz$,

(c) **(4 points)** $\displaystyle\int_\gamma \frac{dz}{2z^2-5z+2}$.

(5) (a) **(5 points)** Find the Laurent series expansion of the function $f(z) = \dfrac{1}{(z-2)(z-3)}$ in the annulus $\{1 < |z-1| < 2\}$.

(b) **(5 points)** Find all the isolated singularities of the following function, and classify them into the three standard types:

$$g(z) = \sin\left(\frac{1}{z}\right) + \frac{\sin(\pi z)}{z(z-1)^3}.$$

(6) Let $a > 0$. Use the method of residues to compute the integral

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^2}.$$

**MTH 632:** Provide complete solutions to **only** 5 of the 6 problems.

(1) Let $E \subseteq \mathbb{R}$.
  (a) **(2 points)** Show that for each $n \in \mathbb{N} \setminus \{0\}$, there exists an open set $U_n$ containing $E$ such that $m(U_n) < m^*(E) + \frac{1}{n}$.
  (b) **(3 points)** Show that there exists a $G_\delta$-set $G$ that contains $E$ such that $m^*(E) = m^*(G)$.
  (c) **(5 points)** If $E$ is a non-measurable set, show that the $G_\delta$-set $G$ in part (b) also satisfies

$$m^*(G \setminus E) > 0.$$

(2) Let $(f_k \colon E \to \mathbb{R})_{k \in \mathbb{N}}$ be a sequence of measurable functions. Let $g \colon E \to \mathbb{R} \cup \{\infty\}$ be given by

$$g(x) = \sup_{k \in \mathbb{N}} \{f_k(x)\}.$$

Prove that $g$ is measurable.

(3) Let $E \subseteq \mathbb{R}$ be a subset of finite measure, and $(f_k \colon E \to \mathbb{R})$ a sequence of measurable functions that converges pointwise a.e. to $f \colon E \to \mathbb{R}$. Show that

$$\lim_{k \to \infty} \int_E \frac{1}{1 + \exp(f_k)} = \int_E \frac{1}{1 + \exp(f)}.$$

Here, $\exp(x) = e^x$.

(4) Construct a sequence of bounded non-negative functions $(f_k)$ that converges pointwise a.e. to 0 on $[0, 1]$ but does not converge to 0 in $L^1([0, 1])$. Justify your answer.

(5) Let $f \colon [-1, 1] \to \mathbb{R}$ be defined by $f(x) = x^2$.
  (a) **(6 points)** Fix integers $0 < k < n$. Let $P = \{a_0, \dots, a_k, a_{k+1}, \dots, a_n\}$ denote a partition of $[-1, 1]$ where

$$\begin{aligned} a_0 &= -1, \\ a_k &\leq 0 < a_{k+1}, \text{ and} \\ a_n &= 1. \end{aligned}$$

  Compute the variation of $f$ with respect to $P$ to show that it is given by

$$V(f, P) = 2 - 2\min(a_k^2, a_{k+1}^2).$$

  (b) **(4 points)** Show that the total variation of $f$ is $TV(f) = 2$.

(6) Let $1 \leq p_1 < p_2 < \infty$.
  (a) **(1 point)** Prove: If $f \in L^{p_2}[0, 1]$, then $f^{p_1} \in L^{\frac{p_2}{p_1}}[0, 1]$.
  (b) **(4 points)** Prove that $L^{p_2}[0, 1] \subseteq L^{p_1}[0, 1]$.
  (c) **(5 points)** Prove that the containment $L^{p_2}[0, 1] \subseteq L^{p_1}[0, 1]$ is strict; that is, find a function in $L^{p_1}[0, 1]$ that is not in $L^{p_2}[0, 1]$.