CENTRAL MICHIGAN UNIVERSITY
DEPARTMENT OF MATHEMATICS
ANALYSIS QUALIFYING EXAM
JANUARY 6, 2022

Each question of this exam is worth 10 points. For questions divided into multiple sub-questions, the points for each part are indicated.

1. Instructions

(1) The exam will have 6 problems from Complex Analysis (MTH 636), and 6 problems from Real Analysis (MTH 632).

(2) You will be required to complete 5 problems from the Complex Analysis (MTH 636) part, and another 5 problems from the Real Analysis (MTH 632) part.

(3) In either section, if you attempt solutions to all 6 problems, then only the first 5 problems will be graded.

(4) Begin each problem on a separate sheet of paper, and clearly write the problem number and your name before beginning your solution.

(5) No calculators or other electronic devices are allowed.

(6) No questions may be asked during the exam. If a problem appears ambiguous to you, interpret it in a way that makes sense to you, but not in a way that makes it trivial.

(7) Give proper mathematical justification for all your statements.

MTH 636: Provide complete solutions to **only** 5 of the 6 problems.

(1) Find a holomorphic function $f\colon \mathbb{C} \to \mathbb{C}$ such that $f(0) = 1$ and whose imaginary part is given by

$$v(x, y) = 6x^2y - 2y^3 + 2y.$$

(2) Show that there are 4 roots of $z^4 + 3(1 + i)z + 2$ inside the circle $|z| = 2$.

(3) Evaluate

$$\int_{\gamma_r} \frac{e^z\, dz}{(z - b)(z^4 - 16)}$$

where γ_r is any circle centered at $(2, 0)$ of positive radius $r < 2$ and $b \in \mathbb{C}$ has negative real part.

(4) Let f be an entire function satisfying $|f(z)| \leq e^{2\operatorname{Re}(z)}$ for each $z \in \mathbb{C}$. Prove that there exists a constant $c \in \mathbb{C}$ for which $f(z) = ce^{2z}$ for each $z \in \mathbb{C}$.

(5) Suppose V is a *bounded* connected open set, and $f\colon \overline{V} \to \mathbb{C}$ is continuous such that f is holomorphic on V. Suppose $c > 0$ such that $|f(z)| \leq c$ for all z in the boundary of V. Prove $|f(z)| \leq c$ for all $z \in V$.

(6) Let f be an entire function whose set of zeroes is infinite. Prove that the set of zeroes is unbounded.

MTH 632: Provide complete solutions to **only** 5 of the 6 problems.

(1) Suppose a function is continuous on $\mathbb{R}$. Must there be an open interval on which the function is either constant or monotone? Justify your response.

(2) Let $h \in L^1([0,1])$ and consider F defined for each $y \in \mathbb{R}$ by

$$F(y) = \int_{[0,1]} h(x)\, \cos^{2021}\left(x^2 y\right) dx,$$

as a Lebesgue integral with respect to the variable x for each fixed $y \in \mathbb{R}$. Prove that F is continuous on $\mathbb{R}$.

(3) Let $E = (0, \infty)$.

(a) **(5 points)** Give a detailed description of a function f that is positive on E and belongs to $L^7(E)$ but not to $L^{10}(E)$.

(b) **(5 points)** Give a detailed description of a function g that is positive on E and belongs to $L^{10}(E)$ but not to $L^7(E)$.

(4) Let f be a function defined on $\mathbb{R}$. If f satisfies the condition that for every rational number r, the set $f^{-1}((r,\infty))$ is measurable, must f be a measurable function? Either prove it or explain a counterexample.

(5) (a) **(5 points)** Give an example of a sequence of functions (f_n) on $[0,1]$ such that $\lim_{n\to\infty} \int |f_n| = 0$, but (f_n) is not uniformly convergent.

(b) **(5 points)** Assume a sequence of integrable functions (f_n) on $\mathbb{R}$ converges uniformly to f. Is it necessarily the case that $\int |f_n - f| \to 0$ as $n \to \infty$?

(6) Consider $L^\infty(E)$, the set of all measurable, essentially bounded functions on a measurable set E. Prove that for every $f \in L^\infty(E)$, $|f| \le ||f||_\infty$ $(a.e.)$ on E, where $||f||_\infty$ is the essential supremum of f.