CENTRAL MICHIGAN UNIVERSITY
DEPARTMENT OF MATHEMATICS
ANALYSIS QUALIFYING EXAM
JANUARY 5, 2023

Each question of this exam is worth 10 points. For questions divided into multiple sub-questions, the points for each part are indicated.

1. Instructions

(1) The exam will have 6 problems from Complex Analysis (MTH 636), and 6 problems from Real Analysis (MTH 632).

(2) You will be required to complete 5 problems from the Complex Analysis (MTH 636) part, and another 5 problems from the Real Analysis (MTH 632) part.

(3) In either section, if you attempt solutions to all 6 problems, then only the first 5 problems will be graded.

(4) Begin each problem on a separate sheet of paper, and clearly write the problem number and your name before beginning your solution.

(5) No calculators or other electronic devices are allowed.

(6) No questions may be asked during the exam. If a problem appears ambiguous to you, interpret it in a way that makes sense to you, but not in a way that makes it trivial.

(7) Give proper mathematical justification for all your statements.

MTH 636: Provide complete solutions to **only** 5 of the 6 problems.

(1) Let
$$u(x,y) = x^2 - y^2 - \frac{y}{x^2+y^2}.$$
Determine whether there exists a holomorphic function f on $\mathbb{C}\backslash\{0\}$ with real part $u(x,y)$. Express $f(z)$ in terms of $z = x+iy$ only (do not leave x and y in your answer).

(2) Let f be an entire function such that $|f(z)| \leq e^{\mathrm{Re}(z)}$ for all $z \in \mathbb{C}$. Prove that $f(z) = ce^z$ for some $c \in \mathbb{C}$.

(3) Evaluate the integral
$$\int_{-\infty}^{\infty} \frac{dx}{1-x^2+x^4}.$$
Express your answer as a real number.

(4) Let f be an entire function, and assume that $\mathrm{Re}\, f(z) \geq 0$ for all $z \in \mathbb{C}$. Prove that f is constant.

(5) Let $f(z)$ be a holomorphic function on an open set $V \subset \mathbb{C}$, and let z_0 and r be such that $z \in V$ whenever $|z - z_0| \leq r$. Prove that
$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{it})dt.$$

(6) Let f be a holomorphic function on the punctured disk $D'(0,r) = \{z : 0 < |z| < r\}$, and suppose that f has a pole of order N at $z = 0$. Prove that there exists $0 < r_0 \leq r$ and a holomorphic function g on $D'(0,r_0)$ such that
$$f(z) = \left(\frac{1}{g(z)}\right)^N$$
for all $z \in D'(0,r_0)$.

MTH 632: Provide complete solutions to **only** 5 of the 6 problems.

(1) Let $(f_k\colon E \to \mathbb{R})_{k\in\mathbb{N}}$ be a sequence of measurable functions. Prove that the following function is measurable:

$$\inf\{f_k\}\colon x \mapsto \inf\{f_k(x) \mid k \in \mathbb{N}\}.$$

(2) Let $E \subseteq \mathbb{R}$ be a subset of finite measure, and $(f_k\colon E \to \mathbb{R})$ a sequence of measurable functions that converge pointwise a.e. to $f\colon E \to \mathbb{R}$. Show that

$$\lim_{k\to\infty} \int_E |\arctan(f_k)| = \int_E |\arctan(f)|.$$

(Recall that arctan is the inverse function of tan.)

(3) Each of the following sequences $(f_n\colon E \to \mathbb{R})$ converges pointwise to some measurable function $f\colon E \to \mathbb{R}$:

(a) $E = [0,1]$ and $f_n(x) = x^n$ for $x \in [0,1]$ and each $n \in \mathbb{N}$,

(b) $E = \mathbb{R}$ and $f_n = n\chi_{[n,n+\frac{1}{n}]}$ for each positive integer n. (Recall that for any $A \subseteq \mathbb{R}$, we define $\chi_A(x)$ to be 1 if $x \in A$ and 0 otherwise.)

For each sequence, determine whether it also

(i) converges uniformly, and if so, to what;

(ii) converges in measure, and if so, to what;

(iii) converges in $L^1(E)$, and if so, to what.

Justify your solutions.

(4) Suppose $1 \le p < \infty$ and q the conjugate of p. Prove for a set $E \subset \mathbb{R}$ of finite measure that if $f \in L^p(E)$ then

$$||f||_p = \max\left\{ \int_E |fg| \;\middle|\; g \in L^q(E) \text{ and } ||g||^q \le 1 \right\}.$$

(5) Let $f\colon [0,1] \to [0,1]$ be differentiable with continuous derivative f'. For each positive integer n, define

$$X_n := \left\{ x \in [0,1] \;\middle|\; |f'(x)| < \frac{1}{n} \right\} \quad \text{and} \quad Y_n := f(X_n).$$

(a) (5 points) Prove that $m(Y_n) \le \frac{1}{n}$.

(b) (5 points) Recall that a **critical point** of f is a point x_0 in the domain of f satisfying $f'(x_0) = 0$; the image $f(x_0)$ is called a **critical value**. Prove that the set of critical values of f has measure zero.

(6) Let $f\colon [a,b] \to \mathbb{R}$ be increasing. Prove that f is measurable.