PH.D. QUALIFYING EXAMINATION-THEORETICAL STATISTICS
Time: 1:00 pm-5:00 pm, January 22, 2021

General Instructions

- There are two equally weighted parts in this exam totaling 100 points. Both Part A (STA584) and Part B (STA684) consist of 5 problems with subproblems, and each part has a possible total of 50 points.
- Write your name on the front of the first page. Do not write your name on any other sheets.
- Begin each question on a new sheet with the question number clearly noted. Write on one side only. When finish, please arrange all pages according to the question order, numbers the pages accordingly, and put them in the envelope.
- You must show all your work and justifications correctly and perfectly to receive full credits, and partial credits may be given for partially correct solutions. If a theorem is employed, you must clearly state the theorem, identify its assumption(s) and conclusion(s), and explain why it is applicable.

Part A

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1. (9 *points*) Find the probability in each of the questions given below.

(1.a) Eighty percent of all vehicles examined at a certain emissions inspection station pass the inspection. Given that at least one of the next three vehicles fails inspection, what is the probability that all three fail? Assume that successive vehicles pass or fail independently.

(1.b) Of all customers purchasing automatic garage-door openers, 75% purchase a chain-driven model. If the store currently has in stock ten chain-driven models and two shaft-driven models, what is the probability that the next 12 customers' requests can all be met from existing stock?

(1.c) A box in a certain supply room contains three 40W light bulbs, four 60W bulbs, and five 75W bulbs. Suppose that two bulbs are randomly selected. Given that at least one of the two selected is not rated 75W, what is the probability that both selected bulbs have the same rating?

2. (11 *points*) Compute the expected value in each of the questions given below.

(2.a) A newsstand has ordered five copies of a certain issue of a photography magazine. Let X be the number of individuals who come in to purchase this magazine. If X has a Poisson distribution with parameter $\lambda=3$, what is the expected number of copies that are sold?

(2.b) A random variable X has the cumulative distribution function

$$F(x) = \begin{cases} 0 & \text{for } x < 1 \\ \frac{1}{2}(x-1)^2 & \text{for } 1 \leq x < 2 \\ 1 & \text{for } x \geq 2. \end{cases}$$

Compute E[X].

(2.c) John and Jane have agreed to meet for lunch between noon (0:00 pm) and 1:00 pm. Denote John's arrival time by X, Jane's by Y, and suppose X and Y are independent with pdf's

$$f(x) = \begin{cases} 3x^2 & \text{for } 0 \leq x \leq 1 \\ 0 & \text{otherwise,} \end{cases}$$

and

$$g(y) = \begin{cases} 2y & \text{for } 0 \leq y \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

What is the expected amount of time that the one who arrives first must wait for the other person?

3. (13 *points*) Find the conditional density, mean, and variance in each of the questions given below.

(3.a) You are given that X and Y are jointly uniform in the region defined by the conditions $0 < x < 2$ and $x^2 < y < 4$. Find $f(x|y=3)$, and compute $E[X|Y=3]$ and $Var(X|Y=3)$.

(3.b) Let X and Y be continuous random variables with joint probability density function given by

$$f(x,y) = \begin{cases} \frac{x}{9} & \text{for } 0 < y < x < 3 \\ 0 & \text{otherwise.} \end{cases}$$

Find $f(x|y>1)$, and compute $E[X|Y>1]$ and $Var(X|Y>1)$.

4. (8 *points*) If the joint probability distribution function of X and Y is given by

$$f(x,y) = c(2x-y)^2.$$

For x=1, 2, 3 and y=1, 2, and c is a constant. Find

(4.a) the value of c.

(4.b) the joint distribution of $U=X+Y$ and $V=2X-Y$.

(4.c) the conditional distribution of $V|U=u$.

(4.d) the moment generating function of $V|U=u$.

5. (9 *points*) If X_1 and X_2 are independent random variables having Poisson distributions with the parameters λ_1 and λ_2 respectively, find the probability distribution of the random variable $Y = X_1 + X_2$ using both *transformation* (*change-of-variable*) and *moment-generating function techniques*. What is the conditional distribution of $X_1|Y = y$?

Part B

1. (10 *points*) Answer both parts.

(1.a) Let $\overline{X}$ and $\overline{Y}$ be the sample means of two independent random samples, each of size n, from the respective distributions N(μ_1, σ^2) and N(μ_2, σ^2), where the common variance is known. Find n such that

$$P\left(|\overline{X} - \overline{Y} - (\mu_1 - \mu_2)| < \sigma/_5 \right) = 0.95$$

(1.b) Let X have an F distribution with parameters r_1 and r_2 then. Show that $Y = 1/X$ has an F distribution with parameters r_2 and r_1.

2. (8 *points*) Let the random variable X_n have a binomial distribution, i.e., $b(n, p)$. Prove that

$$\frac{X_n}{n}\left(1 - \frac{X_n}{n}\right)$$

(2.a) converges in probability to $p(1 - p)$.

(2.b) converges in distribution to $p(1 - p)$.

3. (10 *points*) Suppose $X_1, X_2, \ldots, X_n$ are iid random variables with the pdf

$$f(x;\theta) = \begin{cases} \theta e^{-\theta x} & \text{for } 0 < x < \infty \\ 0 & \text{otherwise.} \end{cases}$$

(3.a) Show that $\sum_{i=1}^{n} X_i$ is a sufficient statistic for θ.

(3.b) Find the mle of θ.

(3.c) Show that $\frac{n-1}{\sum_{i=1}^{n} X_i}$ is the unbiased estimator of θ.

(3.d) Find the MVUE of θ. Explain what theorems are used to find the MVUE.

4. (8 *points*) Let $X_1, X_2, \dots, X_n$ be a random sample from $N(\mu_0, \sigma^2 = \theta)$ distribution, where $0 < \theta < \infty$ and μ_0 is known.

(4.a) Show that the likelihood ratio test of $H_0: \theta = 0$ versus $H_0: \theta \neq 0$ can be based upon the statistic

$$W = \frac{\sum_{i=1}^{n}(X_i - \mu_0)^2}{\theta_0}$$

(4.b) Determine the null distribution of W and give explicitly the rejection rule at a significance level of α.

5. (14 *points*) Answer both parts.

(5.a) Let $Y_1 < Y_2 < \cdots < Y_n$ denote the *n* order statistics based on the random sample $X_1, X_2, \dots, X_n$ from a continuous distribution with the pdf $f(x)$ and support (a, b). Prove that the pdf of Y_k, where $k = 1,2, \dots, n$, is given by

$$g(y_k) = \begin{cases} \dfrac{n!}{(k-1)!\,(n-k)!}[F(y_k)]^{k-1}[1 - F(y_k)]^{n-k} f(y_k) & \text{for } a < y_k < b \\ 0 & \text{otherwise.} \end{cases}$$

(5.b) State and prove the Neyman–Pearson theorem.