General Instruction: This is a closed exam with no any textbook, notes and electronic devices. The exam lasts 180 minutes. Answer all questions. Print your answer and number the pages and number the problems on provided exam papers. Write on one side only. Please use only black pens or pencils.

Notation of Multiple Linear Regression Model:

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \quad X = \begin{bmatrix} 1 & x_{11} & x_{12} & \dots & x_{1k} \\ 1 & x_{21} & x_{22} & \dots & x_{2k} \\ \vdots & \vdots & & \vdots & \\ 1 & x_{n1} & x_{n2} & \dots & x_{nk} \end{bmatrix} \quad \boldsymbol{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_k \end{bmatrix} \quad \boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}$$

$$E(\boldsymbol{\varepsilon}) = \mathbf{0} \qquad \boldsymbol{\varepsilon} = N(\mathbf{0}, \sigma^2 \boldsymbol{I}) \qquad Cov(\boldsymbol{\varepsilon}) = \sigma^2 \boldsymbol{I}$$

1. Given the model, $\boldsymbol{Y} = \boldsymbol{X\beta} + \boldsymbol{\varepsilon}$, where $\boldsymbol{\varepsilon} \sim \boldsymbol{N}(\boldsymbol{0}, \sigma^2 \boldsymbol{I})$, $\boldsymbol{Y}_{n\times 1}$, $\boldsymbol{X}_{n\times(k+1)}$, $\boldsymbol{\beta}_{(k+1)\times 1}$, p=k+1 and $\boldsymbol{\varepsilon}_{n\times 1}$. Prove the following: (20)

   a. $\dfrac{(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})'(\boldsymbol{X'X})(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})}{\sigma^2} \sim \chi^2_p$.

   b. $\hat{\boldsymbol{\beta}}$ is independent of s^2.

   c. $\bar{\boldsymbol{Y}}$ and $\sum_{i=1}^{n} (y_i - \bar{y})^2$ are independent.

2. For model $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$, assume that $\boldsymbol{\varepsilon} \sim \mathbf{N}(\mathbf{0}, \sigma^2 \mathbf{I})$ The statistical hypothesis tests in linear regression model is formed as (20)

   $$\mathbf{H_o}: \mathbf{C}\boldsymbol{\beta} = \boldsymbol{\gamma} \qquad \mathbf{H_a}: \mathbf{C}\boldsymbol{\beta} \neq \boldsymbol{\gamma}$$

   where m is the rank of **C,** a m×(k+1) matrix. Prove the following,

   a. Give the test statistic and the distribution of the statistic

   b. Prove the test statistic in 2.a has the F distribution with degrees of freedom _____ and _____.

3. For model, $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$, where $\boldsymbol{\varepsilon} \sim \mathbf{N}(\mathbf{0}, \sigma^2 \mathbf{I})$, $\mathbf{Y}$ n×1, $\mathbf{X}$ n×p, $\boldsymbol{\beta}$ p×1 and $\boldsymbol{\varepsilon}$ n×1, with patrician (20)

   $\boldsymbol{X} = \left(\boldsymbol{X}^{(1)}\ \boldsymbol{X}^{(2)}\right)$ $\quad \boldsymbol{\beta}' = \left(\boldsymbol{\beta}^{(1)'}\ \boldsymbol{\beta}^{(2)'}\right)$ $\quad \boldsymbol{\beta}^{(1)'} = (\beta_o, \beta_1 \dots \beta_r)$, $\boldsymbol{\beta}^{(2)'} = (\beta_{r+1}, \dots \beta_k)$

   a. For test $\mathbf{H_o}: \hat{\boldsymbol{\beta}}^{(2)} = \mathbf{0}$ $\quad \mathbf{H_a}: \boldsymbol{\beta}^{(2)} \neq \mathbf{0}$ find the $\boldsymbol{C}, \boldsymbol{\gamma}$ and m in the following test statistic. Give the test statistic in the quadratic form notation and the distribution of the test statistic.

   b. Given

   $$\widehat{\boldsymbol{\beta}}_H = \widehat{\boldsymbol{\beta}} + (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{C}'[\boldsymbol{C}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{C}']^{-1}(\boldsymbol{C}\widehat{\boldsymbol{\beta}} - \boldsymbol{\gamma}) \qquad \text{where} \qquad \widehat{\boldsymbol{\beta}}_H = \hat{\boldsymbol{\beta}}^{(2)}$$

Prove

$$(\boldsymbol{y} - \boldsymbol{X}\widehat{\boldsymbol{\beta}}_{\boldsymbol{H}})'(\boldsymbol{y} - \boldsymbol{X}\widehat{\boldsymbol{\beta}}_{\boldsymbol{H}}) - (\boldsymbol{y} - \boldsymbol{X}\widehat{\boldsymbol{\beta}})'(\boldsymbol{y} - \boldsymbol{X}\widehat{\boldsymbol{\beta}}) = (\boldsymbol{C}\widehat{\boldsymbol{\beta}} - \boldsymbol{\gamma})'[\boldsymbol{C}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{C}']^{-1}(\boldsymbol{C}\widehat{\boldsymbol{\beta}} - \boldsymbol{\gamma})$$

c. Give the SSE, SSE_F and SSE_R in their quadratic form notation and prove the test statistic in part 3.a equals

$$\frac{SSE_F - SSE_R}{SSE}\left(\frac{n-k-1}{m}\right).$$

4. Given the Growth Model $\quad \overline{Y} = X\beta + \overline{\varepsilon} \quad$ where (20)

$$Y = \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1M} \\ y_{21} & y_{22} & \cdots & y_{2M} \\ \vdots & \vdots & & \vdots \\ y_{m1} & y_{m2} & \cdots & y_{mM} \end{bmatrix} \quad \overline{Y} = \begin{bmatrix} \overline{y}_1 \\ \overline{y}_2 \\ \vdots \\ \overline{y}_m \end{bmatrix} \quad \overline{\varepsilon} = \begin{bmatrix} \overline{\varepsilon}_1 \\ \overline{\varepsilon}_2 \\ \vdots \\ \overline{\varepsilon}_m \end{bmatrix} \quad \varepsilon = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \cdots & \varepsilon_{1M} \\ \varepsilon_{21} & \varepsilon_{22} & \cdots & \varepsilon_{2M} \\ \vdots & \vdots & & \vdots \\ \varepsilon_{m1} & \varepsilon_{m2} & \cdots & \varepsilon_{mM} \end{bmatrix}$$

$$\overline{Y}_i = \frac{1}{M}\sum_{i=1}^{M} y_{it} \qquad \overline{\varepsilon}_i = \frac{1}{M}\sum_{i=1}^{M} \varepsilon_{it} \qquad t = 1, 2, \ldots M \qquad i = 1, 2, \ldots m$$

$X \quad m \times p \qquad cov(\varepsilon_t) = \Omega \qquad \varepsilon_t s$ are independent.

$\overline{Y}_i s$ are independent.

a. find the $cov(\overline{\varepsilon})$

b. find the estimated generalized least square estimate of β.

c. prove that the b_{EGLS} obtained in b) is unbiased estimate of β.

5. For Model (1) (20)

a. Define the Studentized Residuals $e_i^* \quad i = 1, 2, \ldots n.$
b. Show that Studentized Residual in part a. has Student's t distribution with the degrees of freedom *n-k-2.*